(Relaxed) Product Structures of Graphs and Hypergraphs

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Abstract

In this thesis, we investigate graphs and hypergraphs that have (relaxed) product structures.

In the class of graphs, we discuss in detail *RSP-relations*, a relaxation of relations fulfilling the square property and therefore of the product relation σ, that identifies the copies of the prime factors of a graph w.r.t. the Cartesian product. For $K_{2,3}$-free graphs finest RSP-relations can be computed in polynomial-time. In general, however, they are not unique and their number may even grow exponentially. Explicit constructions of such relations in complete and complete bipartite graphs are given.

Furthermore, we establish the close connection of (*well-behaved*) RSP-relations to (quasi-)covers of graphs and equitable partitions. Thereby, we characterize the existence of non-trivial RSP-relations by means of the existence of spanning subgraphs that yield quasi-covers of the graph under investigation. We show, how equitable partitions on the vertex set of a graph G arise in a natural way from well-behaved RSP-relations on $E(G)$. These partitions in turn give rise to quotient graphs that have rich product structure even if G itself is prime. This product structure of the quotient graph is still retained even for RSP-relations that are not well-behaved. Furthermore, we will see that a (finest) RSP-relation of a product graph can be obtained easily from (finest) RSP-relations on the prime factors w.r.t. certain products and in what manner the quotient graphs of the product w.r.t such an RSP-relation result from the quotient graphs of the factors and the respective product.

In addition, we examine relations on the edge sets of *hyper*graphs that satisfy the grid property, the hypergraph analog of the square property. We introduce the *strong* and the *relaxed* grid property as variations of the grid property, the latter generalizing the relaxed square property. We thereby show, that many, although not all results for graphs and the (relaxed) square property can be transferred to hypergraphs. Similar to the graph case, any equivalence relation R on the edge set of a hypergraph H that satisfies the relaxed grid property induces a partition of the vertex set of H which in turn determines quotient hypergraphs that have non-trivial product structures. Besides, we introduce the notion of

(Cartesian) hypergraph bundles, the analog of (Cartesian) graph bundles and point out the connection between the grid property and hypergraph bundles.

Finally, we show that every connected thin hypergraph H has a unique prime factorization with respect to the normal and strong (hypergraph) product. Both products coincide with the usual strong *graph* product whenever H is a graph. We introduce the notion of the Cartesian skeleton of hypergraphs as a natural generalization of the Cartesian skeleton of graphs and prove that it is uniquely defined for thin hypergraphs. Moreover, we show that the Cartesian skeleton of thin hypergraphs and its PFD w.r.t. the strong and the normal product can be computed in polynomial time.

Contents

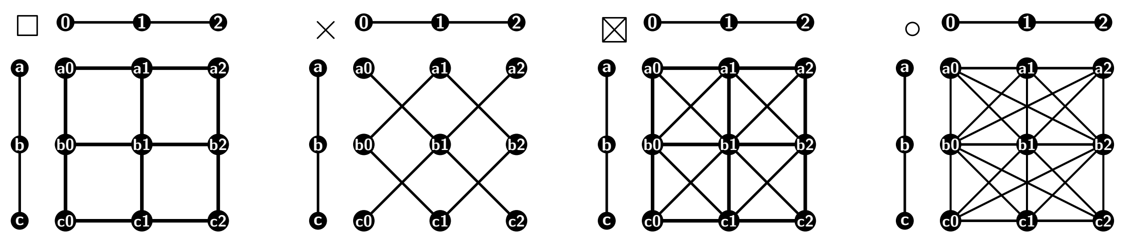
Chapter 1

Introduction

Products are a common way in mathematics of constructing larger objects from smaller building blocks. It is of key interest then to understand the structure of a large object by decomposing it into its prime building blocks.

Graph products are natural structures in discrete mathematics [26, 54] that arise in a variety of different contexts, from computer science [2, 29, 31] and computational engineering [48, 49] to theoretical biology [20, 21, 8, 61, 63]. They can be constructed in many different ways. For example, different constructions arise depending on whether loops are considered or not. As shown in [42], it is possible to define 256 different products of graphs such that the vertex set of a product graph is the Cartsian product of the vertex sets of its factors and adjacency in the product graph depends only on the adjacency properties in the factors. However, there are only four "standard graph products" that preserve the salient structure of their factors and behave in an algebraically reasonable way: The *Cartesian product* $\Box$, the *direct product* $\times$, the *strong product* $\boxtimes$, and the *lexicographic product* $\circ$, see Figure 1.1. Their structural features have been studied extensively over the last decades. It is well known how many of the important graph invariants propagate under product formation. Results on uniqueness of prime factor decomposition (PFD) are available for the standard graph products [58, 62, 11, 52, 40] and efficient algorithms have been devised to decompose graph products into their prime factors [15, 3, 41, 16, 31, 17]. Several monographs cover the topic in substantial detail and serve as standard references [43, 44, 28].

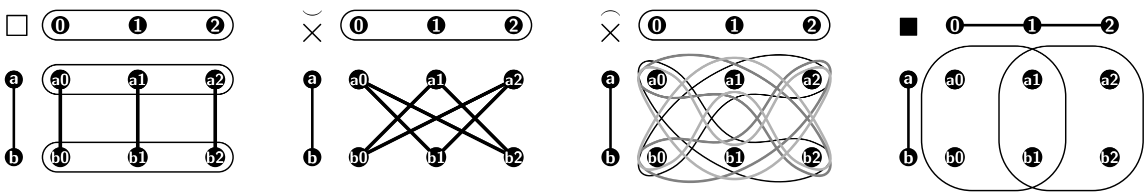
In contrast, very little is known about product structures of hypergraphs, even though hypergraphs have become increasingly important models of network structures. Most of the constructions of hypergraph products can be seen as generalizations of the four standard graph products. Thereby, it is possible to find several non-equivalent generalizations of graph products, especially the direct and the strong graph product [37]. The most widely studied variant, the so-called *square product*, however, is an exception. Literature on hypergraph



(a) $G_1 \Box G_2$. (b) $G_1 \times G_2$. (c) $G_1 \boxtimes G_2$. (d) $G_1 \circ G_2$.

Fig. 1.1: The four standard products. In all cases $V(G_1 \circledast G_2) = V(G_1) \times V(G_2)$. **(a)** Two vertices (x_1, x_2), (y_1, y_2) are adjacent in the *Cartesian product* $G_1 \Box G_2$ if (i) $(x_1, y_1) \in E(G_1)$ and $x_2 = y_2$, or (ii) $(x_2, y_2) \in E(G_2)$ and $x_1 = y_1$. **(b)** They are adjacent in the *direct product* $G_1 \times G_2$ if (iii) $(x_1, y_1) \in E(G_1)$ and $(x_2, y_2) \in E(G_2)$. **(c)** The *strong product* $G_1 \boxtimes G_2$ has edge set $E(G_1 \Box G_2) \cup E(G_1 \times G_2)$. **(d)** (x_1, x_2), (y_1, y_2) are adjacent in the *lexicographic product* $G_1 \circ G_2$ if they satisfy (ii) or if (iv) $(x_1, y_1) \in E(G_1)$.

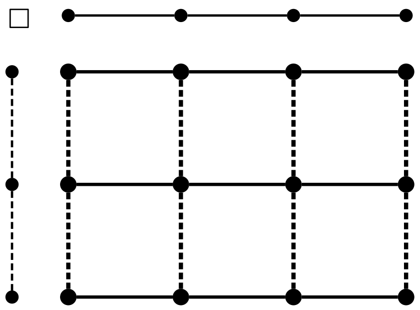
products is mostly concerned with the propagation of invariants. Unique PFD results are known for the Cartesian product [39], the lexicographic product and the so called *co-strong product* [23], and the square product [10] of hypergraphs. A first polynomial time algorithm to compute the PFD of connected undirected hypergraphs w.r.t. Cartesian product is given in [6]. A recent survey [37] gathers the existing literature, focusing on the basic properties of the various hypergraph products and their mutual relationships.



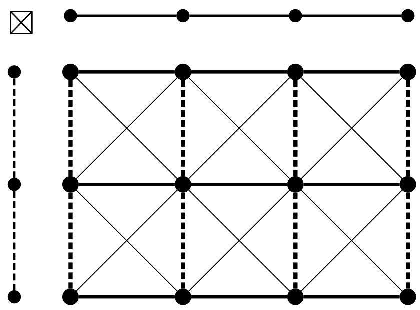
(a) $H_1 \Box H_2$. (b) $H_1 \breve{\times} H_2$. (c) $H_1 \hat{\times} H_2$. (d) $H_1 \blacksquare H_2'$.

Fig. 1.2: Some Hypergraph products. **(a)** The Cartesian product can be generalized straight forward from graphs to hypergraphs. **(b)** and **(c)** Two different hypergraph products generalizing the direct graph product. **(d)** The *square product* of two graphs is no longer a graph but a 4-uniform hypergraph.

Prime factorizations of graphs and hypergraphs are closely related to so-called *product relations* on their edge sets. For instance, modern proofs of PFD theorems for the Cartesian graph product rely on characterizations of the product relation σ on the edge set of the given graph [47]. The key property of σ is that connected components of the subgraphs induced by the classes of σ are precisely the *layers*, i.e., $(e, f) \in \sigma$ if and only if the edges e and f belong to copies of the same (Cartesian) prime factor [58, 28], see Figure 1.3(a). Classical results in the theory of graph products establish that σ can be derived from other, easily computable, relations on the edge set. It is shown in [47], that σ is just the convex hull of some locally defined relation δ, that satisfy the so-called *square property*, a restrictive condition on unique chordless squares, see Figure 1.4 for an example. Following this, analogous results can be obtained for the Cartesian product of hypergraphs [53], where the role of chordless squares are replaced by diagonal-free grids, generalizing the square property to the so-called *grid property*.



(a) $P_2 \Box P_3$.

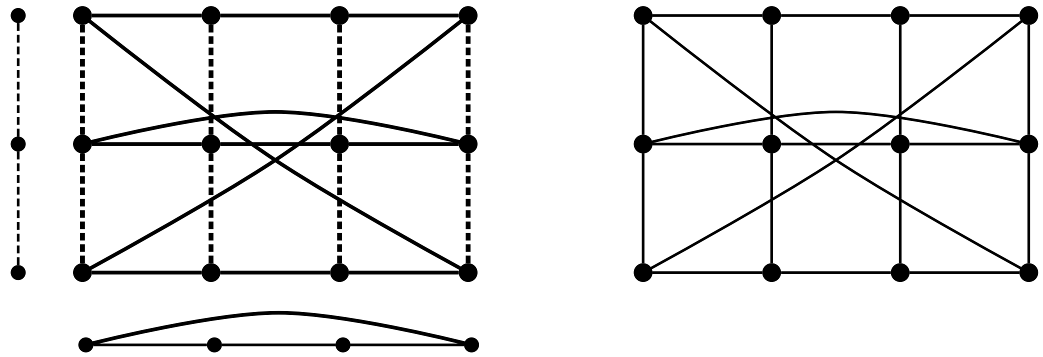


(b) $P_2 \boxtimes P_3$.

Fig. 1.3: **(a)** The product relation σ on the edge set of a Cartesian product graph, whose classes are depicted by dashed, resp. drawn-through lines, reflects the layers, i.e. isomorphic copies of the factors, w.r.t. its PFD. **(b)** Product Relation σ on the Cartesian skeleton of a strong product graph with the same factors as in (a). Equivalence classes of σ are depicted by dashed, resp. drawn-through lines. Non-Cartesian edges are indicated by thin lines.

The basic idea underlying *relaxed product structures* is to construct generalizations of the product relation that still retain some, but not all properties of the fibers. A *relaxed product*, therefore, is a (hyper)graph admitting a non-trivial relaxed product relation. As an example, in the class of graphs, a slight modification of the relation δ and the square property, that still retains their salient properties, turned out to play a fundamental role for the characterization of *Cartesian graph bundles* [66], a common generalization of both Cartesian products [28] and covering graphs [1]. Hence, graph bundles can be viewed as relaxed products. By further relaxations of certain conditions of the square property, that is by omitting requirements for unique and chordless squares, one obtains equivalence relations R that have the *relaxed square property.* For hypergraphs, relaxations of the grid property in an analogous way may lead to the *relaxed grid property.* However, analogous constructions as graph bundles for hypergraphs have not yet found consideration in the literature. Generalizations of product relations provide an avenue to product-like graphs that is fundamentally different from the "approximate product graphs" studied in [33, 34, 31, 35] in terms of coverings by small factorizable subgraphs.

In the class of graphs, the Cartesian product is closely related to the strong product and plays a central role in the PFD of strong product graphs, see Figure 1.3(b). In several approaches [16, 27, 31], the key idea for the prime factorization of a strong product graph G is to find a subgraph $\mathbb{S}(G)$ of G with special properties, the so-called *Cartesian skeleton*,



(a) The relation δ^*.

(b) Product relation $\sigma = \mathcal{C}(\delta)$.

Fig. 1.4: **(a)** The equivalence relation δ^*, i.e., the tranistive closure of the relation δ, on the edges set of a graph G with two equivalence classes, depicted by dashed, resp. drawn-through lines. In fact, G is a Cartesian graph bundle. The dashed edges correspond to the isomorphic copies of the fiber. **(b)** The convex hull of the relation δ, i.e., the product relation σ has only one equivalence class, since indeed, G is prime w.r.t. Cartesian product.

that is then decomposed with respect to the *Cartesian* product. Afterwards, one constructs the prime factors of G using the information of the PFD of $\mathbb{S}(G)$. Since it is known, that Cartesian hypergraph products have unique PFD in the class of connected hypergraphs, this seems to be a promising ansatz for finding the prime factors of a hypergraph w.r.t. the strong and normal hypergraph product.

Within this work, we will examine systematically certain equivalence relations on the edge set of a graph, so-called *RSP-relations*, that generalize the product relation σ. Furthermore, we generalize the main statements to hypergraphs and the relaxed grid property. Thereby, we introduce the notion of Cartesian hypergraph bundles as the hypergraph analog of Cartesian graph bundles. Finally, we consider prime factorization of thin hypergraphs w.r.t. the strong and the normal product, generalizing a well-known approach for graphs and introduce thereby the *Cartesian skeleton* of a hypergraph.

This thesis is divided into two parts. In the first part, we consider graphs and equivalence relations on their edge sets that characterize graphs as relaxed products. Basic concepts in graph theory, a short introduction to graph products as well as terminology used in context of equivalence relations within this thesis will be provided in **Chapter 2**.

In **Chapter 3**, we shortly introduce the (unique) square property and the closely connected relation δ. Then, we turn to the relaxed square property, resp. *(well-behaved)* RSP-relations. We show, that (some of) the structural features provided by the (unique) square property are still retained under this relaxation and discuss difficulties in finding a finest RSP-relation.

In **Chapter 4**, we establish the close connection between (well-behaved) RSP-relations, (quasi-)covers and equitable partitions. We give an alternative characterization of graphs that admit a non-trivial RSP-relation in terms of quasi-covers and we will see, how equitable partitions of the vertex set of a graph can be constructed from well-behaved RSP-relations on its edge set.

The quotient graphs w.r.t. these partitions provide a natural product structure, even when the graph itself is prime, as we will see in **Chapter 5**. Also if we neglect well-behaviour, the product structure of these quotients is retained. We will see, how RSP-relations and the structure of the respective quotient graphs of a Cartesian, direct or strong product graph can be transfered from its factors.

Part I is based on the following articles: M. Hellmuth, L. Ostermeier, and P. F. Stadler. Unique square property, equitable partitions, and product-like graphs. *Discrete Mathematics*, 320(0):92 – 103, 2014. And M. Hellmuth, T. Marc, L. Ostermeier, and P. F. Stadler. The relaxed square property. *Australasian Journal of Combinatorics*, accepted for publication, 2015.

In the second part of this thesis, we examine certain product structures in hypergraphs. Basic concepts in the theory of hypergraphs will be provided in **Chapter 6**. Moreover, we will give a short overview about hypergraph products. To be more precise, we will consider the Cartesian product as well as two different hypergraph products that both generalize the strong graph product, the normal product and strong product.

Then, in **Chapter 7**, we examine the grid property, the hypergraph analog of the square property, and its modifications, that plays a crucial role in the theory of Cartesian hypergraph products. We thereby generalize the main statements that were made in Part I for graphs to hypergraphs. We will introduce Cartesian hypergraph bundles as a generalization of Cartesian graph bundles and examine its connection to the grid property.

Finally, we turn to the normal product and the strong product of hypergraphs in **Chapter 8**, both generalizing the strong graph product. Following the key idea of factorization of graphs w.r.t. the strong product, we introduce the Cartesian skeleton $\mathbb{S}(H)$, a partial hypergraph of the hypergraph H, and use this to obtain PFD uniqueness results of the normal and the strong hypergraph product. For this purpose, the 2-*section* $[H]_2$ of a hypergraph H, the graph defined on $V(H)$ reflecting the adjacencies in H, turns out to be a useful tool. It therefore enables the application of well known results for graphs to hypergraphs. On this basis, an algorithm for computing the PFD of thin hypergraphs is given.

Part II is based on the following articles: L. Ostermeier, and P. F. Stadler. The Grid Property and Product-Like Hypergraphs. *Graphs and Combinatorics*, 31(3):757 – 770, 2015. And M. Hellmuth, L. Ostermeier, and M. Noll. Strong products of hypergraphs: Unique prime factorization theorems and algorithms. *Discr. Appl. Math.*, 171:60–71, 2014.

Part I

RELAXED GRAPH PRODUCTS

Chapter 2

Basics I

Within this chapter, we provide basic definitions, that will be used in this thesis. We begin with general concepts in graph theory, mainly following the notation in [9]. We proceed with a short overview about graph products, where we give the definition and basic properties of the Cartesian, the strong and the direct product. Finally, we consider equivalence relations on the edge set and partitions of the vertex set of a graph.

2.1 Graphs

An *(undirected) graph* is a pair $G = (V(G), E(G))$ of *vertex set* $V(G)$ and a family $E(G)$ consisting of unordered pairs of elements of $V(G)$, the *edges* of G. To avoid ambiguity, we always assume that $V(G) \cap E(G) = \emptyset$. An edge $e = [u, v]$ is said to *join* the vertices u and v. u and v are then called *endpoints* of e. An edge e with identical endpoints, $e = [u, u]$, is called *loop*. A graph G is *simple* if $E(G)$ is a set, i.e., there are no multiple edges, and contains no loops. With $\mathcal{L}G$ we denote the graph which emerges from G by adding a loop to every vertex $v \in G$. If G is a graph that is not simple, we denote with $\mathcal{N}(G)$ the underlying simple graph, i.e., $V(\mathcal{N}(G)) = V(G)$ and $E(\mathcal{N}(G))$ is the set that contains all edges of G exactly once and with all loops deleted.

An edge e is *incident* to a vertex v, if v is an endpoint of e. Two edges are said to be *adjacent* if they have a common endpoint. Two vertices are *adjacent* if they are joined by an edge. A subset of the vertex set $V(G)$ whose vertices are pairwise not adjacent is called *independent*. A graph G is *totally disconnected* if $V(G)$ is an independent set. The set of vertices that are adjacent to a vertex v is called *(open) neighborhood* of v, denoted with $N_G(v)$. A graph G is called *complete graph* if its vertices are pairwise adjacent. It will be denoted by $K_{|V(G)|}$, where $|X|$ is the cardinality of a set X. G is a *complete bipartite graph* if its vertex set can be partitioned into two independent sets, $V(G) = X\dot{\cup}Y$, such that any vertex from X is adjacent to any other vertex from Y. It will be denoted by $K_{|X|,|Y|}$.

The *degree* $deg_G(v)$ of a vertex v in G is the number of edges incident to v. If G is a simple graph, we have $deg_G(v) = |N_G(v)|$. With $\delta(G)$ and $\Delta(G)$ we denote the *minimum degree* and *maximum degree* of G, respectively. That is, $\delta(G) = \min_{v\in V(G)} deg_G(v)$ and $\Delta(G) = \max_{v\in V(G)} deg_G(v)$

Paths, Cycles. A *walk* in a graph G is a sequence $P_{v_0,v_k} = (v_0, v_1, \ldots, v_k)$, where $[v_{i-1}, v_i] \in E(G)$ for all $i = 1, \ldots, k$. We may also write $P_{v_0,v_k} = (e_1, e_2 \ldots, e_k)$, with $e_i \in E(G)$, where $e_i = [v_{i-1}, v_i]$. The walk P_{v_0,v_k} is said to *join* the vertices v_0 and v_k. A *path* is a walk where the vertices $v_0, \ldots, v_k$ and edges are all distinct. The *length of a path* is the number of edges contained in the path. A path of length k will often be denoted with P_k. A *k-cycle* is a sequence $C = (v_0, v_1, \ldots, v_{k-1}, v_0)$, such that $P_{v_0,v_{k-1}}$ is a path. A *triangle* is a 3-cycle. A 4-cycle $Sq = (v_0, v_1, v_2, v_3, v_0)$ is often called *square*, denoted with $v_0 - v_1 - v_2 - v_3$. An edge $d = [v_i, v_{i+2}]$, subscripts taken modulo 4, is called *diagonal* or *chord* of the square Sq. If there is no such edge, Sq is called *chordless* or *diagonal free*. Any two adjacent edges in a square Sq are said to *span* Sq. Two nonadjacent edges in a square are called *opposite* edges.

The *distance* $d_G(v, v')$ between two vertices v, v' of G is the length of a shortest path joining them. We set $d_G(v, v') = \infty$ if there is no such path. A graph G is called *connected*, if any two vertices are joined by a path.

Subgraphs. A graph H is a subgraph of G, $H \subseteq G$, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A subgraph H is an *induced subgraph* of G if $x, y \in V(H)$ and $[x, y] \in E(G)$ implies $[x, y] \in E(H)$. H is called *spanning subgraph* if $V(H) = V(G)$. If none of the subgraphs H of G is isomorphic to a graph K, we say that G is *K-free*. A maximal connected subgraph of G is called *connected component* of G.

A connected subgraph $H \subseteq G$ is called *convex*, if all shortest paths in G between two vertices in H are also contained in H. We say that H is *k-convex* if for any pair of vertices $u, v \in V(H)$ of distance $d_G(u, v) \leq k$, the set of all shortest paths from u to v in G is also contained in H. Note, a 1-convex subgraph is an induced subgraph and convexity is the same as ∞-convexity. For general H, i.e., if H is not necessarily connected, we call H (k-)convex, if all of its connected components are (k-)convex.

Homomorphisms and Covering Constructions. For two graphs G and H a *homomorphism* from G into H, written as $f : G \to H$, is a mapping $f : V(G) \to V(H)$ such that $f(u)$ and $f(v)$ are adjacent in H whenever u and v are adjacent in G. A mapping $f : V(G) \to V(H)$ is a *weak homomorphism* if adjacent vertices are mapped either to adjacent vertices or the same vertex. Weak homomorphisms are also known as *graph maps*.

A homomorphism f that is bijective is called an *isomorphism* if holds $[f(u), f(v)] \in E(H)$ if and only if $[u, v] \in E(G)$. We say, G and H are *isomorphic*, in symbols $G \cong H$ if there exists an isomorphism between them. An isomorphism from a graph G onto itself is called *automorphism.* $Aut(G)$ denotes the set of all automorphisms of a graph G. Homomorphisms of graphs have been widely investigated, for example in [30].

A homomorphism $f : G \to H$ between two graphs G and H is called *locally surjective* if $f(N_G(u)) = N_H(f(u))$ for all vertices $u \in V(G)$, i.e., if $f_{|N_G(u)} : N_G(u) \to N_H(f(u))$ is a surjection. Analogously, f is called *locally bijective* if for all vertices $u \in V(G)$ it holds that $f(N_G(u)) = N_H(f(u))$ and $|f(N_G(u))| = |N_H(f(u))|$, i.e., $f_{|N_G(u)} : N_G(u) \to N_H(f(u))$ is a bijection. Notice, a locally surjective homomorphism $f : G \to H$ is already globally surjective if H is connected. If there exists a locally surjective homomorphism $f : G \to H$, we call G a *quasi-cover* of H. Locally surjective homomorphisms are also known as role colorings [13]. A locally bijective homomorphism is called a *covering map*. G is a *(graph) cover* or *covering graph* of H if there exists a covering map from G to H, in which case we say that G *covers* H. $|V(H)|$ is then a multiple of $|V(G)|$, i.e., $|V(H)| = k|V(G)|$. H is then referred to as *k-fold cover* of G. Moreover, every covering map $f : H \to G$ satisfies $|f^{-1}(u)| = k$ for all $u \in V(G)$ [19]. The *universal cover* of a graph G is the (possibly infinite) tree that covers G. It is unique up to isomorphism. For more detailed information about locally constrained homomorphisms and graph cover we refer to [18, 19].

Let $f : G \to H$ be a weak homomorphism. By abuse of language, the subgraph of G that is induced by the vertex set $\{x \in V(G) \mid f(x) = v\}$ for $v \in V(H)$ is denoted with $f^{-1}(v)$. The subgraph of G induced by the vertex set $\{x \in V(G) \mid f(x) \in e\}$ for $e \in E(H)$ is denoted with $f^{-1}(e)$. Note, $f^{-1}(v)$ and $f^{-1}(e)$ actually refers to sets. However, it will be clear from the context what is meant.

Directed Graphs, Weighted Graphs. At some point, we also might consider directed and directed edge weighted graphs. A *directed graph* is a pair $\vec{G} = (V(\vec{G}), E(\vec{G}))$ of vertex set $V(\vec{G})$ and a family $E(\vec{G})$ consisting of *ordered* pairs of elements of $V(\vec{G})$, the *arcs* of G. Arcs are also called directed edges and we denote them with round brackets, $(u, v) \in E(\vec{G})$. Note, an arc $(u, v) \in E(\vec{G})$ joins vertex u to vertex v, not vice versa. All definitions made for undirected graph can also be extended to directed graphs. For a directed graph $\vec{G}$, the *underlying undirected graph* G, has vertex set $V(\vec{G}) = V(G)$ and edges $[u, v] \in E(G)$ iff $(u, v) \in E(\vec{G})$.

A directed *edge weighted* graph is a directed graph together with a *weight function* $w : E(\vec{G}) \to \mathbb{R}$ assigning a *weight* $w(u, v)$ to each edge $(u, v) \in E(\vec{G})$. Formally, one can extend w to a function $w' : V(\vec{G}) \times V(\vec{G}) \to \mathbb{R}$. by setting $w'(u, v) = w(u, v)$ iff $(u, v) \in E(\vec{G})$ and $w'(u, v) = 0$ otherwise. In that manner, any (non-simple) directed graph can be viewed as (simple) edge weighted directed graph with weight function $w' : V(\vec{G}) \times V(\vec{G}) \to \mathbb{N}$, such that $w'(u, v)$ denotes the number of edges connecting u to v and $E(\vec{G}) = \{(u, v) \mid w'(u, v) > 0\}$, and vice versa.

Analogously, edge weights are defined for undirected graphs. Note, it then holds $w(u, v) = w(v, u)$ for all vertices u, v.

For a weighted graph G, we denote with $\mathcal{N}(G)$ the underlying simple graph with weights omitted and we set $\mathcal{N}(\overrightarrow{G}) := \mathcal{N}(G)$.

Remark 2.1. *Unless otherwise explicitly stated, we consider finite, connected, simple graphs.*

2.2 Graph Products

Graph products are defined as graphs whose vertex set is the Cartesian product of the vertex sets of its factors. Requiring that the adjacency of vertices in the product depends only on the adjacency in the factors, it was shown already in 1975 that there are 256 possible products, of which exactly twenty are associative [42]. Four of them are known as the *standard graph products* [43, 28]: the *Cartesian* product $\Box$, the *direct* product $\times$, the *strong* product $\boxtimes$, and the *lexicographic* product $\circ$. These are the only associative simple products that depend on the structure of both factors and for which at least one of the projections is a weak homomorphism, only the first three ones are commutative. Their structural features have been studied extensively over the last decades. It is well known how many of the important graph invariants propagate under product formation. A comprehensive survey about graph products in general can be found in the *Handbook of Product Graphs* [28].

Within this thesis, we will focus mainly on the Cartesian product, the most fundamental one. However, we will also consider some aspects concerning direct and strong product.

The Cartesian Product

Let G and H be graphs. The *Cartesian product* $G \Box H$ has vertex set $V(G \Box H) = V(G) \times V(H)$; two vertices (g_1, h_1), (g_2, h_2) are adjacent in $G \Box H$ if one of the following conditions is satisfied:

(i) $[g_1, g_2] \in E(G)$ and $h_1 = h_2$, or

(ii) $[h_1, h_2] \in E(H)$ and $g_1 = g_2$.

Figure 2.1 shows the Cartesian product of a cycle of length 5 and an edge.

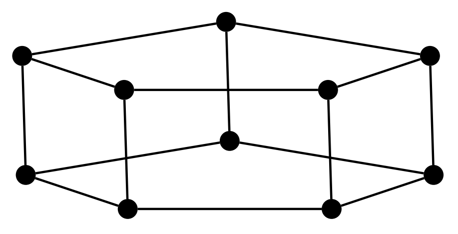


Fig. 2.1: Cartesian Product $K_2 \Box C_5$.

The Cartesian product is associative and commutative. It is distributive w.r.t. the disjoint union of graphs. The one-vertex graph K_1 serves as a unit, i.e. $G \Box K_1 \cong G$ holds for all graphs G. The Cartesian product of two graphs is connected if and only if both factors are connected. It is simple if and only if both factors are simple. [44]

The mapping $p_i : V(\Box_{i=1}^n G_i) \to V(G_i)$ defined by $p_i(v) = v_i$ for $v = (v_1, v_2, \ldots, v_n)$ is called *projection* on the i-th factor of G. The *G_i-layer through w*, G_i^w, is the induced subgraph of G with vertex set $V(G_i^w) = \{v \in V(G) \mid p_j(v) = w_j, \text{ for all } j \neq i\}$. It is isomorphic to G_i. The projections $p_i : \Box_{i=1}^n G_i \to G_i$ are weak homomorphisms for all $i = 1, \ldots, n$.

Prime Factor Decomposition. A graph G is called *prime* w.r.t. Cartesian product if $G = G_1 \Box G_2$ implies $G_1 \cong K_1$ or $G_2 \cong K_1$. A factorization $G = \Box_{i=1}^n G_i$ is called *prime factor decomposition* (PFD) w.r.t. Cartesian product if all G_i are prime.

The PFD of disconnected graphs w.r.t. the Cartesian product need not be unique. As an example the following identity is given in [43]

$$(K_1 + K_2 + K_2^2)\Box(K_1 + K_2^3) = (K_1 + K_2^2 + K_2^4)\Box(K_1 + K_2)$$

However, Sabidussi [58] and later Vizing [62] showed the following:

Theorem 2.2 ([58, 62])**.** *Every connected graph has a unique representation as a Cartesian product of prime graphs, up to isomorphisms and the order of the factors.*

A series of polynomial time algorithms that compute the PFD of connected graphs w.r.t. Cartesian product can be found in the literature, see [15, 14, 3, 45, 64].

Cartesian Product of Weighted Graphs. Cartesian products generalize in a natural way to directed edge-weighted graphs (with loops allowed). Their Cartesian product $G \Box H$ has the edge weights

$$m((g_1, h_1), (g_2, h_2)) = \begin{cases} m_G(g_1, g_2), & \text{iff } h_1 = h_2 \text{ and } g_1 \neq g_2 \\ m_H(h_1, h_2), & \text{iff } g_1 = g_2 \text{ and } h_1 \neq h_2 \\ m_G(g_1, g_2) + m_H(h_1, h_2), & \text{iff } g_1 = g_2 \text{ and } h_1 = h_2 \\ 0, & \text{otherwise} \end{cases} \tag{2.1}$$

where $m_G(g_1, g_2)$ and $m_H(h_1, h_2)$ denotes the edge weight of the arcs (g_1, g_2) in G and (h_1, h_2) in H, respectively. The absence of such an arc is equivalent to $m_X(x_1, x_2) = 0$ for $X \in \{G, H\}$. The Cartesian product of the underlying undirected and unweighted graphs is obtained by ignoring the directions and weights in the product graph.

Cartesian Graph Bundles. (Cartesian) Graph bundles are a common generalization of both Cartesian products and covering graphs. They were first studied in [56].

A graph G is a *(Cartesian) graph bundle* if there are two graphs F, the *fiber*, and B, the *base graph*, and a graph map $p : G \to B$ such that holds: $p^{-1}(v) \cong F$ for each vertex $v \in V(B)$, and $p^{-1}(e) \cong K_2 \Box F$ for each edge $e \in E(B)$. Figure 2.2 shows a Cartesian graph bundle over base graph K_3 and with fiber K_2. The triple (G, p, B) is called a presentation of

G as a Cartesian graph bundle, or *bundle presentation* for short. If $G = \Box_{i=1}^{n} G_i$ is a product, then (G, p_j, G_j) is a bundle presentation of G with fiber $\Box_{i=1, i \neq j}^{n} G_i$ for all $1 \leq j \leq n$.

Conversely, for given graphs B, F one can construct a graph bundle G over the base graph B with fiber F as follows: Define a mapping $\alpha : V(B) \times V(B) \to Aut(G)$ such that $\alpha(u, v) = \alpha^{-1}(v, u)$ and $\alpha(u, u) = id$ holds for all $u \in V(B)$. For brevity, we will write α_{uv} instead of $\alpha(u, v)$. The vertex set of the graph bundle G is given by $V(G) = V(B) \times V(F)$. Two vertices $(b_1, f_1), (b_1, f_2)$ are adjacent in G if one of the following conditions is satisfied:

(i) $b_1 = b_2$ and $[f_1, f_2] \in E(F)$, or

(ii) $[b_1, b_2] \in E(B)$ and $f_2 = \alpha_{b_1 b_2}(f_2)$.

For a bundle presentation (G, p, B), an edge $e \in E(G)$ is called *degenerate* if $p(e)$ is a single vertex, otherwise it is *non-degenerate*. The *fundamental factorization* of G partitions G into edge-disjoint spanning subgraphs H and $\tilde{G}$ with $E(G) = E(H) \dot{\cup} E(\tilde{G})$, such that H consists of isomorphic copies of the fiber F and $\tilde{G}$ contains exactly the non-degenerate edges of G. The mapping $p : \tilde{G} \to B$ is a covering projection. Moreover, covering graphs are precisely graph bundles with totally disconnected fiber.

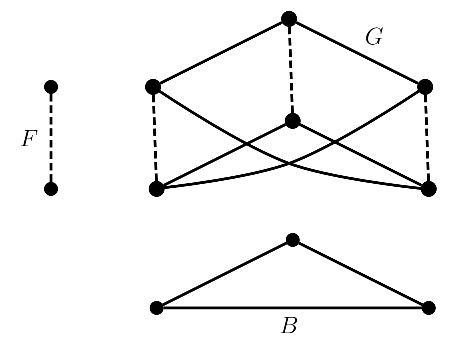


Fig. 2.2: Cartesian graph bundle G over base graph $B = K_3$ with fiber $F = K_2$. The non-degenerate edges are highlighted with drawn-through lines, the degenerate edges, that belong to the copies of the fiber, are depicted with dashed lines.

Also other standard graph products can be generalized to graph bundles. However, unless otherwise stated, the term *graph bundle* always refers to Cartesian graph bundle.

The Strong Product and the Direct Product

As the Cartesian product, the *strong product* $G \boxtimes H$ and the *direct product* $G \times H$ have vertex set $V(G \boxtimes H) = V(G \times H) = V(G) \times V(H)$; two vertices (g_1, h_1), (g_2, h_2) are adjacent in $G \boxtimes H$ if one of the following conditions is satisfied:

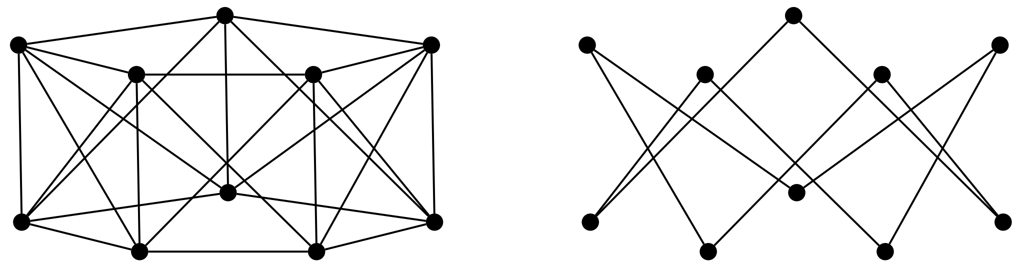
(i) $[g_1, g_2] \in E(G)$ and $h_1 = h_2$, or

(ii) $[h_1, h_2] \in E(H)$ and $g_1 = g_2$, or

(iii) $[g_1, g_2] \in E(G)$ and $[h_1, h_2] \in E(H)$.

(g_1, h_1), (g_2, h_2) are adjacent in $G \times H$ if they satisfy only Condition (iii).

Figure 2.3 shows the strong product and the direct product of a cycle of length 5 and an edge. Note, by definition it holds $E(G \boxtimes H) = E(G \Box H) \cup E(G \times H)$ for all graphs G, H. Hence, any strong product $G \boxtimes H$ has the Cartesian product $G \Box H$ as a spanning subgraph. Edges of $E(G \boxtimes H)$ that are contained in $E(G \Box H)$, i.e., that satisfy condition (i) or (ii) of the definition of the strong product, are called *Cartesian*, the others *non-Cartesian* edges.



(a) Strong product $C_5 \boxtimes K_2$.　(b) Direct product $C_5 \times K_2$.

Fig. 2.3: Strong and direct product of the graphs C_5 and K_2.

The strong product and the direct product are both associative, commutative and distributive w.r.t. the disjoint union of graphs. While the one-vertex graph K_1 serves as a unit for the strong product, i.e. $G \boxtimes K_1 \cong G$ holds for all graphs G, the direct product has no unit in the class of simple graphs (without loops). However, in the class of simple graphs with loops allowed, the one-vertex graph with a loop, $\mathcal{L}K_1$ is a unit for the direct product, that is, $G \times \mathcal{L}K_1 \cong G$ holds for all graphs G with or without loops. The strong product of two graphs is connected if and only if both factors are connected. The direct product of two graphs with at least one edge is connected if and only if both factors are connected and at least one of them is non-bipartite. The direct, resp. strong product of two graphs is simple if and only if both factors are simple. [28]

Similarly to the Cartesian product, disconnected graphs may have non-unique PFD w.r.t. the strong product and the direct product. The following result concerning unique PFD of the strong product is due to W. Dörfler and W. Imrich.

Theorem 2.3 ([12])**.** *Every connected graph has unique prime factor decomposition with respect to the strong product.*

For the direct product, there is no unique PFD in the class of connected simple graphs. However, in [52], McKenzie showed the following:

Theorem 2.4 ([52])**.** *Every finite, non bipartite connected graph (possibly with loops) has unique prime factor decomposition with respect to the direct product in the class of graphs with loops allowed.*

The PFD w.r.t both, the strong and the direct product can be computed in polynomial time, see [16, 41].

2.3 Relations, Partitions, Quotient Graphs

Let X be a set. A set $\mathcal{P} = \{X_1, \dots, X_n\}$ of disjoint subsets of X is called *partition* of X if $\dot{\bigcup}_{i=1}^{n} X_i = X$. The elements of $\mathcal{P}$ are called *classes* of the partition.

A *(binary) relation* on X is a subset $R \subseteq X \times X$. A relation R on X is called

(R) *reflexive* if $(x, x) \in R$ for all $x \in X$,

(S) *symmetric* if $(x, y) \in R$ implies $(y, x) \in R$ for all $x, y \in X$,

(A) *antisymmetric* if $(x, y) \in R$ and $(y, x) \in R$ implies $x = y$ for all $x, y \in X$,

(T) *transitive* if $(x, y) \in R$ and $(y, z) \in R$ implies $(x, z) \in R$ for all $x, y, z \in X$.

A *partial order* on X is a a reflexive, antisymmetric and transitive relation on X. A relation R on X that is reflexive, symmetric and transitive is an *equivalence relation*. An equivalence relation R on X partitions the set X into *equivalence classes*: two elements of X are in relation R if and only if they belong to the same equivalence class. Two equivalence classes are either disjoint or the same, the (disjoint) union of all equivalence classes is the set X itself. The equivalence classes of an equivalence relation R will be denoted by Greek letters, $\varphi \subseteq X$. We will furthermore write $\varphi \sqsubseteq R$ for mean that φ is an equivalence class of R. The *complement* $\overline{\varphi}$ of an R-class φ is defined as $\overline{\varphi} := X \setminus \varphi = \bigcup_{\psi \neq \varphi, \psi \sqsubseteq R} \psi$.

A relation Q is finer than a relation R while the relation R is coarser than Q if $(x, y) \in Q$ implies $(x, y) \in R$, i.e, $Q \subseteq R$. In other words, for each class ϑ of R there is a collection $\{\chi | \chi \subseteq \vartheta\}$ of Q-classes, whose union equals ϑ. Equivalently, for all $\varphi \sqsubseteq Q$ and $\psi \sqsubseteq R$ we have either $\varphi \subseteq \psi$ or $\varphi \cap \psi = \emptyset$.

Relations on the Edge Set of a Graph

We will consider equivalence relations R on the edge set $E(G)$ of a graph G.

For an equivalence class $\varphi \sqsubseteq R$, an edge e is called φ-edge if $e \in \varphi$. For a given equivalence class $\varphi \sqsubseteq R$ and a vertex $u \in V(G)$ we denote the set of neighbors of u that are incident to u via a φ-edge by $N_\varphi(u)$, i.e.,

$$N_\varphi(u) := \{v \in V(G) \mid [u, v] \in \varphi\}.$$

The subgraph G_φ has vertex set $V(G)$ and edge set φ. The connected component of G_φ containing vertex $x \in V(G)$ is called *φ-layer through* x, denoted by G_φ^x. Analogously, the subgraphs $G_{\overline{\varphi}}$ and $G_{\overline{\varphi}}^x$ are defined. For a φ-layer G_φ^x and a vertex $y \in V(G)$ holds either $y \in V(G)$ and thus $G_\varphi^x = G_\varphi^y$ or $G_\varphi^x \cap G_\varphi^y = \emptyset$. Two φ-layers G_φ^x, G_φ^y are said to be *adjacent*, if there exists a $\overline{\varphi}$-edge $[x', y']$ with $x' \in V(G_\varphi^x)$ and $y' \in V(G_\varphi^y)$.

Convex Relations. Let R be an equivalence relation on the edge set $E(G)$ of a graph G. Suppose, R has equivalence classes $\varphi_i, i \in I$. We say R is *convex* if for any $K \subseteq I$ the subgraph G_χ with $\chi = \bigcup_{i \in K} \varphi_i$ is convex. The *convex hull* $\mathcal{C}(R)$ of a relation R is the minimal convex equivalence relation on $E(G)$ that contains R. Well known examples for convex equivalence relations are the product relation w.r.t. Cartesian product, see below, and the transitive closure θ^* of the so-called *Djoković-Winkler relation* θ, that is defined as follows: Two edges $[a, b]$ and $[x, y]$ are in relation θ if

$$d(a, x) + d(b, y) \neq d(a, y) + d(b, x).$$

An equivalence class $\varphi \sqsubseteq R$ is called k*-convex* if G_φ is k-convex. We say R is k*-convex* if each equivalence class of R is k-convex. R is called *weakly* k*-convex* if at least one equivalence class of R is k-convex. As it turns out, weakly 2-convex equivalence relations play a prominent role for recognizing Cartesian graph bundles [46, 66, 65].

Product Relations. An equivalence relation R on the edge set $E(G)$ of a Cartesian product $G = \Box_{i=1}^n G_i$ of (not necessarily prime) graphs G_i is a *product relation* if $e\ R\ f$ holds if and only if there exists a $j \in \{1, \ldots, n\}$ such that $|p_j(e)| = |p_j(f)| = 2$. The product relation according to a PFD $G = \Box_{i=1}^n G_i$ is often denoted with σ.

It was shown by Feder in [14], that σ can be represented as $\sigma = (\theta \cup \tau)^*$, where two edges $e = [x, y]$ and $f = [y, z]$ are in relation τ if y is the only common neighbor of x and z. In other words, $e \tau f$ if e and f are adjacent and there is no square in G containing both edges.

In another approach [47], it was shown by Imrich and Žerovnik that any convex equivalence relation on $E(G)$ that satisfies the so-called *square property* is a product relation. Moreover they proved that σ is nothing else but the convex hull of a certain relation δ, $\sigma = \mathcal{C}(\delta)$. We will consider the square property and the relation δ more detailed in Section 3.1.

Quotient Graphs and Equitable Partitions

Let G be a graph and $\mathcal{P}$ be a partition of $V(G)$. The (undirected) *quotient graph* $G/\mathcal{P}$ has as its vertex set $\mathcal{P}$, i.e., the classes of the partition. There is an edge $[A, B]$ for $A, B \in \mathcal{P}$ if and only if there are vertices $a \in A$ and $b \in B$ such that $[a, b] \in E(G)$. Note that there is a loop $[A, A]$ unless the class A of $\mathcal{P}$ is an independent set.

A partition $\mathcal{P}$ of the vertex set $V(G)$ of a graph G is *equitable* if, for all (not necessarily distinct) classes $A, B \in \mathcal{P}$ every vertex $x \in A$ has the same number

$$m_{AB} := |N_G(x) \cap B|$$

of neighbors in B. The $|\mathcal{P}| \times |\mathcal{P}|$ matrix $\mathbf{M} = \{m_{AB}\}$ indexed by the classes of $\mathcal{P}$ is known as the *partition degree matrix*. The unique equitable partition of a graph G that has the fewest number of classes is also called *degree partition*. The degree matrix corresponding to this coarsest equitable partition is called *degree refinement matrix*. It is uniquely determined, once the classes of the partition are ordered in a unique way. Equitable partitions of graphs were originally introduced as a means of simplifying the computation of graph spectra [59] and walks on graphs [24].

The *directed weighted quotient graph* $\overrightarrow{G/\mathcal{P}}$ of a graph G w.r.t an equitable partition $\mathcal{P}$ has vertex set $V(\overrightarrow{G/\mathcal{P}}) = \mathcal{P}$ and directed edges (A, B) from A to B with weight m_{AB} iff $m_{AB} \geq 1$. Note that $\overrightarrow{G/\mathcal{P}}$ has loops whenever $m_{AA} \geq 1$. By construction, $m_{AB} \geq 1$ implies $m_{BA} \geq 1$. Hence $\overrightarrow{G/\mathcal{P}}$ has a well-defined underlying undirected and unweighted graph, which obviously coincides with $G/\mathcal{P}$. The underlying simple graph, obtained by also omitting the loops, will be denoted by $\mathcal{N}(\overrightarrow{G/\mathcal{P}}) = \mathcal{N}(G/\mathcal{P})$.

From Edge Partitions to Vertex Partitions

We start from an equivalence relation R on $E(G)$. Let $\varphi \sqsubseteq R$.

By construction, the set

$$\mathcal{P}^R_\varphi := \left\{ V(G^x_\varphi) \mid x \in V(G) \right\}$$

is a partition of $V(G)$ for every $\varphi \sqsubseteq R$. The quotient graph $G/\mathcal{P}^R_\varphi$ has as its vertex sets the connected components G^x_φ and edges $[G^x_\varphi, G^y_\varphi]$ if and only if there are $x' \in V(G^x_\varphi)$ and $y' \in V(G^y_\varphi)$ with $[x', y'] \in E(G)$.

The projection $p_\varphi : G \to G/\mathcal{P}^R_\varphi$ defined by $x \mapsto G^x_\varphi$ is a graph map. If $[x, y] \in \varphi$ then $y \in V(G^x_\varphi)$ and hence $G^x_\varphi = G^y_\varphi$. Thus, we have a loop in the quotient graph $G/\mathcal{P}^R_\varphi$ for every $V(G^x_\varphi) \neq \{x\}$. Edges that do not form a loop in $G/\mathcal{P}^R_\varphi$ thus arise only from $[x, y] \in E \setminus \varphi$.

In the following we will be interested in particular in the complements of R-classes, i.e., in $\overline{\varphi} := E \setminus \varphi$ with corresponding subgraphs $G_{\overline{\varphi}}$ and connected components $G^x_{\overline{\varphi}}$ for a given $x \in V(G)$. For later reference we note following simple

Observation 2.5. *It is the case that $y \in V(G^x_{\overline{\varphi}})$ if and only if there is a path $P := (x = x_0, x_1, \ldots, x_k = y)$ from x to y such that $[x_i, x_{i+1}] \notin \varphi$ for all $0 \leq i \leq k - 1$.*

Just like $\mathcal{P}^R_\varphi$, the set

$$\mathcal{P}^R_{\overline{\varphi}} := \left\{ V(G^x_{\overline{\varphi}}) \mid x \in V(G) \right\}$$

is a partition of $V(G)$ for every $\varphi \sqsubseteq R$. To see this, we note that $x \in V(G^x_{\overline{\varphi}})$ holds for all $x \in V(G)$. Thus, $P \neq \emptyset$ for all $P \in \mathcal{P}^R_{\overline{\varphi}}$ and $\bigcup_{P \in \mathcal{P}^R_{\overline{\varphi}}} P = V(G)$. Furthermore, $V(G^x_{\overline{\varphi}}) \cap V(G^y_{\overline{\varphi}}) \neq \emptyset$ if and only if x and y are in same connected component w.r.t. $\overline{\varphi}$, i.e., if and only if $V(G^x_{\overline{\varphi}}) = V(G^y_{\overline{\varphi}})$. Note, Graham and Winkler showed in [25] that the equivalence relation θ^* induces a canonical isometric embedding of a graph G into a Cartesian product $\Box_{\varphi \sqsubseteq \theta^*} G_\varphi / \mathcal{P}^{\theta^*}_{\overline{\varphi}}$. Moreover, Feder showed that if we choose the equivalence relation $R = (\theta \cup \tau)^*$ then $G \cong \Box_{\varphi \sqsubseteq R} G_\varphi / \mathcal{P}^R_{\overline{\varphi}}$ and thus, $(\theta \cup \tau)^*$ is the product relation σ, see [14].

We furthermore will need the intersections

$$V_R(x) := \bigcap_{\varphi \sqsubseteq R} V(G^x_{\overline{\varphi}}) \, .$$

These sets form the classes of the common refinement of the partitions $\mathcal{P}^R_{\overline{\varphi}}$, $\varphi \sqsubseteq R$, i.e.,

$$\mathcal{P}^R := \left\{ \bigcap_{\varphi \sqsubseteq R} V(G_{\overline{\varphi}}(x)) \mid x \in V(G) \right\} = \{V_R(x) \mid x \in V(G)\}$$

is again a partition of $V(G)$.

Lemma 2.6. *Let Q and R be two equivalence relations on $E(G)$ so that Q is finer than R. Then $V_R(x) \subseteq V_Q(x)$.*

Proof. Consider two equivalence classes $\varphi, \psi \sqsubseteq Q$. From $\overline{\varphi \cup \psi} = \overline{\varphi} \cap \overline{\psi}$ we observe that $G_{\overline{\varphi \cup \psi}}$ is a subgraph of both $G_{\overline{\varphi}}$ and $G_{\overline{\psi}}$. This remains true for the connected components containing a given vertex $x \in V$, and hence

$$V(G^x_{\overline{\varphi \cup \psi}}) \subseteq V(G^x_{\overline{\varphi}}) \cap V(G^x_{\overline{\psi}}) \, .$$

Using this observation we compute

$$V_R(x) = \bigcap_{\vartheta \sqsubseteq R} V\left(G^x_{\overline{\vartheta}}\right) = \bigcap_{\vartheta \sqsubseteq R} V\left(G^x_{\overline{\bigcup_{\chi \subseteq \vartheta} \chi}}\right) \subseteq \bigcap_{\vartheta \sqsubseteq R} \bigcap_{\chi \subseteq \vartheta} V\left(G^x_{\overline{\chi}}\right) = \bigcap_{\chi \sqsubseteq Q} V\left(G^x_{\overline{\chi}}\right) = V_Q(x).$$

□

Thus, a coarser equivalence relation R on $E(G)$ leads to smaller sets $V_R(x)$, and hence to a finer partition $\mathcal{P}^R$ of the vertex set.

Chapter 3

From Square Property to Relaxed Square Property

Within this chapter, we will consider relations on the edge set of a graph that satisfy restrictive conditions on (chordless) squares. In Section 3.1, we will give a short overview about the "(unique) square property" and a closely connected relation δ, that turned out to play a crucial role in the theory of Cartesian graph products and graph bundles. The relation δ, resp. δ^* was introduced by Sabidussi in [58] and then further characterized by Feigenbaum et al. in [15]. The square property was explicitly defined by Imrich and Žerovnik in [47]. A mild generalization, the unique square property, that still retains the structural features of the square property, resp. relation δ^*, was introduced by Zmazek and Žerovnik in [66].

In Section 3.2, we examine a further relaxation of the unique square property, the so-called "relaxed square property". As it turns out, this generalization still retains (some of) the salient properties of the (unique) square property.

3.1 Square Property and Unique Square Property

Definition 3.1 (Relation δ)**.** *Two edges $e, f \in E(G)$ are in the* relation δ*, written $e \;\delta\; f$, if one of the following conditions is satisfied:*

(i) e and f are opposite edges of a chordless square.

(ii) e and f are adjacent and there is no chordless square containing e and f.

(iii) $e = f$.

Obviously, the relation δ is reflexive and symmetric. Its transitive closure, δ^*, is therefore an equivalence relation. It is the smallest equivalence relation containing δ. The relation δ, resp. δ^* was introduced by Sabidussi in [58]. There, he used δ^* as a start relation to build up a tower of equivalence relations on the edge set of a graph G, that finally results in the product relation σ according to the unique PFD of G. In [15], Feigenbaum et al. examined this relation further. In [47], it was shown by Imrich and Žerovnik, that the product relation σ is just the the smallest convex equivalence relation containing δ, i.e., its convex hull, $\sigma = \mathcal{C}(\delta)$.

Definition 3.2 ((Unique) Square Property)**.** *An equivalence relation R on $E(G)$ has the* unique square property *if it satisfies*

(S1) Any two adjacent edges e and f from distinct equivalence classes span a unique chordless square with opposite edges in the same equivalence class of R.

R has the square property *if it satisfies in addition*

(S2) The opposite edges of any chordless square belong to the same equivalence class.

If an equivalence relation R has the square property then adjacent edges of different classes span exactly one chordless square. In contrast, if R has the unique square property then adjacent edges of different classes may span more than one chordless square. In this case one and only one of these squares has opposite edges in the same equivalence class.

Relations with the unique square property need not satisfy the square property. On the other hand, from the definition it is clear, that every equivalence relation R on $E(G)$ that has the square property also has the unique square property.

The following observation has been used implicitly e.g. in [46, 66]. It demonstrates the close connection between the relation δ and the square property.

Proposition 3.3. *An equivalence relation R on $E(G)$ has the square property if and only if $\delta \subseteq R$.*

Proof. Let R be an equivalence relation on $E(G)$ and $\delta \subseteq R$. Then Condition (i) in the definition of δ directly implies Condition (S2). Let e, f be two adjacent edges and suppose $(e, f) \notin R$. Then there must exist a square containing both edges, otherwise, by condition (ii), $(e, f) \in \delta \subseteq R$, a contradiction. Let this square consist of edges e, f, e', f' such that e' is opposite edge to e and f' is opposite edge to f. Then condition (i) implies $(e, e'), (f, f') \in \delta \subseteq R$. Suppose e, f are contained in another square consisting of edges e, f, e'', f'' such that e'' is opposite edge to e and f'' is opposite edge to f. Then there is also a square consisting of edges e', f', f'', e'' such that e'' is opposite edge to f' and f'' is opposite edge to e'. Again condition (i) implies $(e, e''), (f, f'') \in \delta \subseteq R$ as well as $(e'', f'), (f'', e') \in \delta \subseteq R$. Finally, by transitivity it follows $(e, f) \in R$, a contradiction. Thus, R has the square property.

Let R be an equivalence relation on $E(G)$ with the square property. We have to show that $e\ \delta\ f$ implies $e\ R\ f$ for all edges e, f. Suppose first, $e\ \delta\ f$ such that e and f are not adjacent. Hence, either Condition (i) or (iii) is fulfilled which immediately implies $e\ R\ f$. Now, let e and f be adjacent and suppose for contraposition that $e \not R f$. Thus, by condition $(S1)$ there is a chordless square spanned by e and f and therefore e and f do not satisfy condition (ii). Hence, $e \not\delta f$ which completes the proof. $\square$

The transitive closure δ^* of δ is therefore the finest equivalence relation on $E(G)$ that has the square property. Furthermore, an equivalence relation R has the square property if and only if its classes are unions of equivalence classes of δ^*. Therefore, if R has the square property and $R \subseteq S$, then the coarser equivalence relation S also has the square property.

The square property was explicitly introduced by Imrich and Žerovnik in [47]. There, the authors showed that structural properties gained for the relation δ^* in [15], also apply for relations that satisfy the square property, i.e., coarsenings of the relation δ^*. Moreover, any convex equivalence relation with square property is a product relation and thus a coarsening of the relation σ [47].

The square property also applies to the theory of graph bundles: Any weakly 2-convex equivalence relation R on the edge set of a graph G that satisfies the square property induces a fundamental factorization of G. If $\varphi \sqsubseteq R$ is 2-convex, the φ-layers are precisely the isomorphic copies of the fiber [46].

However, the square property fails to detect *all* graph bundles, namely those with triangles in the base graph. As an example consider the graph bundle G in Figure 2.2. The equivalence relation on $E(G)$, that separates non-degenerate from degenerate edges does not satisfy the square property. However, it has the unique square property. The unique square property was introduced by Zmazek and Žerovnik [66]. It was shown, that any equivalence relation on the edge set of a connected graph G, that separates degenerate and non-degenerate edges w.r.t. a bundle presentation of G over arbitrary simple base graph, satisfies the unique square property.

In contrast to the square property, there is no unique finest equivalence relation that has the unique square property. Moreover, if an equivalence relation R satisfies the unique square property, it is still possible that there exists a coarser equivalence relation $S \supset R$ that does not have the unique square property, as shown by the example in Figure 3.1.

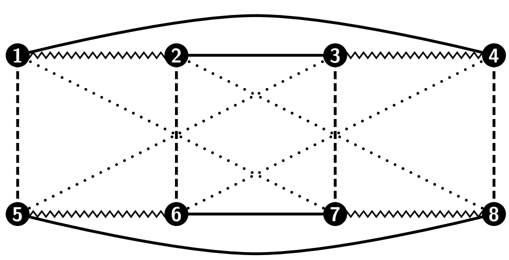


Fig. 3.1: The equivalence relation Q on the edge set $E(G)$ of the “diagonalized cube” G has the four equivalence classes $\varphi_1, \varphi_2, \varphi_3$ and φ_4 depicted by solid, zigzag, dotted and dashed edges, respectively. One easily checks that Q has the unique square property. The relation R with classes $\psi_1 = \varphi_1 \cup \varphi_2$ and $\psi_2 = \varphi_3 \cup \varphi_4$, however, does not have the unique square property, because the edges $[1,5]$ and $[1,2]$ span *two* squares $(1,5,6,2)$ and $(1,5,6,4)$ with opposite edges belonging to the same class.

3.2 Relaxed Square Property

Definition 3.4 (Relaxed Square Property)**.** *Let R be an equivalence relation on the edge set $E(G)$ of a connected graph G. We say R has the* relaxed square property *if any two adjacent edges e, f of G that belong to distinct equivalence classes of R span a square with opposite edges in the same equivalence class of R.*

An equivalence relation R on $E(G)$ with the relaxed square property will be called an *RSP-relation* for short. In contrast to the more familiar (unique) square property, we do not require there that squares spanned by incident edges that belong to different equivalence classes are unique or chordless.

The relaxed square property is preserved under coarse grainings:

Lemma 3.5. *Let R be an RSP-relation on the edge set E of a connected graph $G = (V, E)$. If S is a coarser equivalence relation, $R \subseteq S$, then S is also an RSP-relation on E.*

For later reference we record the following technical result:

Lemma 3.6. *Let R be an RSP-relation on the edge set E of a connected graph $G = (V, E)$ and φ be an equivalence class of R. Moreover, let S be the equivalence relation on the edge set $E \setminus \varphi$ of the spanning subgraph $G' = (V, E \setminus \varphi)$ of G that retains all equivalence classes $\psi \neq \varphi$ of R. Then S is an RSP-relation.*

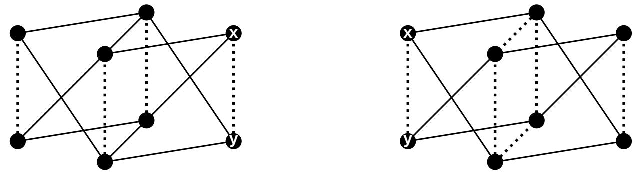
Proof. Let e, f be adjacent edges in $E(G')$ such that $(e, f) \notin S$, say $e \in \psi, f \in \psi'$, $\varphi \neq \psi, \psi' \sqsubseteq S \subseteq R$. By construction, $e, f \in E(G)$ and $(e, f) \notin R$. Thus, there exists a square with edges e, f, e', f' such that e, e' and f, f' are opposite edges and $e' \in \psi$ as well as $f' \in \psi'$. Hence, $e', f' \in E(G')$ and thus the assertion follows. $\square$

The RSP-relation S on the spanning subgraph, as defined in Lemma 3.6, need not be a finest RSP-relation, although R might be a finest one. Consider the right graph in Figure 3.3. If S consists only of the class $\overline{\varphi}$ that is highlighted by the drawn-through edges, then the spanning subgraph $H = (V(G), E(G) \setminus \varphi)$ is the Cartesian graph product of a path on three vertices and an edge. The finest RSP-relation on $E(H)$ is thus the product relation σ w.r.t. the unique $\Box$-PFD of H with two equivalence classes.

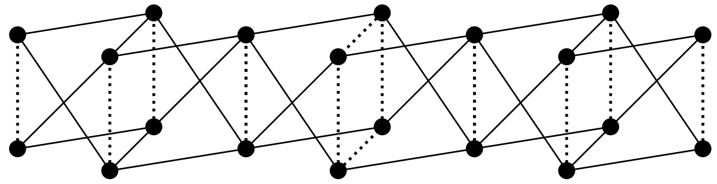
Finest RSP-Relations. As the examples in Figures 3.2, 3.3 and 3.4 show, there is no unique finest RSP-relation for a given graph G. Even more, the number of such finest relations on a graph can grow exponentially as the example in Figure 3.2 shows.

Figure 3.3 gives an example that finest RSP-relations on the edge set $E(G)$ of a graph G need not have the same number of equivalence classes. Moreover, the quotient graphs that are induced by these relations need not be isomorphic, as the example in Fig. 3.4 shows.

We next discuss the relationship of (finest) RSP-relations with relations on the edge set that play a role in the theory of product graphs and graph bundles. By definition, the relation



(a)



(b)

Fig. 3.2: In Fig. (a) two isomorphic graphs with two non-equivalent finest RSP-relations are shown. Each RSP-relation has two equivalence classes, highlighted by dashed and solid edges. By stepwisely identifying the vertices marked with x and y, resp., one obtains a chain of graphs G, see Fig. (b). For each subgraph that is a copy of the graph above, a finest RSP-relation can be determined independently of the remaining parts of the graph G. Hence, with an increasing number of vertices of such chains G the number of finest RSP-relations is growing exponentially.

τ (see Section 2.3) is contained in any RSP-relation. On the other hand, it is easy to see that square property implies relaxed square property, hence δ^* is an upper bound for any *finest* RSP-relation. Thus, if R is a finest RSP-relation, we have $\tau^* \subseteq R \subseteq \delta^*$. However, it is possible to improve these bounds, as we will see in the following.

Definition 3.7 (Relation δ_0)**.** *Two edges $e, f \in E(G)$ are in the* relation δ_0*, $e\ \delta_0\ f$, if one of the following conditions is satisfied:*

(i) e and f are opposite edges of a square.

(ii) e and f are adjacent and there is no square containing e and f, i.e. $(e, f) \in \tau$.

(iii) $e = f$.

The relation δ_0 is reflexive and symmetric. Its transitive closure, denoted with δ_0^*, is therefore an equivalence relation.

Proposition 3.8. *Let G be a connected $K_{2,3}$-free graph and R an equivalence relation on $E(G)$. Then R has the relaxed square property if and only if $\delta_0 \subseteq R$.*

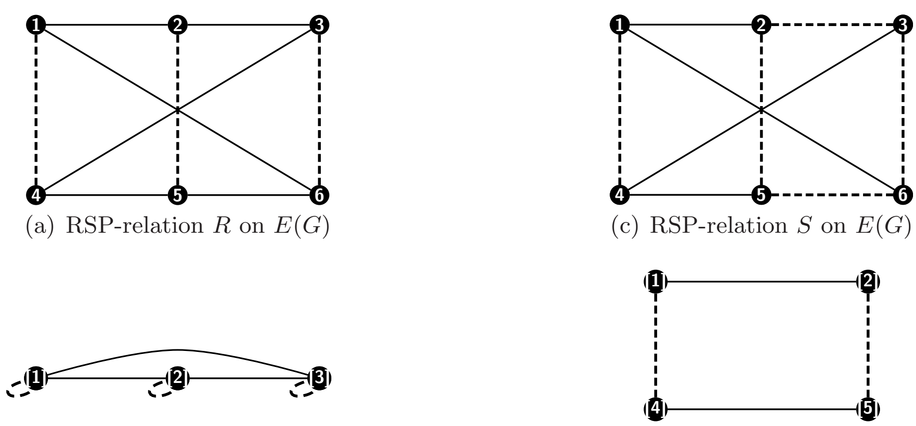
Proof. It is easy to see, that δ_0^* has the relaxed square property and moreover, that any equivalence relation containing δ_0 has the relaxed square property.

Let R be an RSP-relation on the edge set of a connected $K_{2,3}$-free graph G. Notice, if G contains no $K_{2,3}$ then any pair of adjacent edges of G span at most one square. Let e, f

(a) RSP-relation R on $E(G)$

(b) RSP-relation S on $E(G)$

Fig. 3.3: The two panels show two distinct finest RSP-relations (a) R and (b) S on the edge set of a graph G with different number of equivalence classes. We leave it to the reader to verify that the relations, whose equivalence classes are indicated by different line styles, indeed satisfy the relaxed square property. The RSP-relation in (a) has three and that in (b) has two equivalence classes. It remains to show, that both RSP-relations are finest ones.
(a) For all equivalence classes there is a vertex that is incident to exactly one edge of each class. Lemma 3.5 implies that R is a finest RSP-relation.
(b) Assume the relation is not finest. The equivalence class indicated by the dashed edges cannot be subdivided further since this would lead to vertices that are not met by each of the two or more subclasses, thus contradicting Lemma 3.5. The equivalence class depicted by drawn-through edges is isomorphic to a Cartesian product $P_3 \Box K_2$. Using Lemma 3.6, the only possible split would be the Product relation on this subgraph, i.e., with classes $\psi_1 = \{[1,2],[3,4],[5,6]\}$ and $\psi_2 = \{[1,4],[1,6],[2,3],[2,5]\}$. But then there is no square with opposite edges in the same equivalence classes spanned by the edges $[2,3]$ and $[3,5]$, again a contradiction.



(a) RSP-relation R on $E(G)$

(c) RSP-relation S on $E(G)$

(b) G/P^R

(d) G/P^S

Fig. 3.4: Two distinct RSP-relations R and S on the edge set of the same graph G and the quotient graphs induced by these relations. Their coarsest common refinement does not have the relaxed square property. Moreover, the quotient graphs induced by these relations are not isomorphic.

be two edges in G such that $(e, f) \in \delta_0$. We have to show that this implies $(e, f) \in R$. If $e = f$, then $(e, f) \in R$ is trivially fulfilled since R is an equivalence relation. If e and f are not adjacent, they have to be opposite edges of a square. Let g be an edge of this square that is adjacent to both edges e and f. If e and g are not in relation R, by the relaxed square property, they span some square with opposite edges in the same equivalence class. Since G contains no $K_{2,3}$, this square is unique, thus $(e, f) \in R$. Assume now, $(e, g) \in R$. If e and f are not in the same equivalence class of R, we can conclude that also f and g are in distinct equivalence classes, since R is an equivalence relation. Thus, by the relaxed square property, f and g span a square with opposite edges in the same equivalence class and as G is $K_{2,3}$-free, this square has to be unique, which implies $(e, f) \in R$, a contradiction. Now let e and f be two adjacent edges and suppose for contraposition $(e, f) \notin R$. Hence, e and f have to span a square. Thus, condition (ii) in the definition of δ_0 is not satisfied, hence, $(e, f) \notin \delta_0$. In summary, we can conclude $\delta_0 \subseteq R$. □

Proposition 3.8 implies that there is a uniquely determined finest RSP-relation, namely the relation δ_0^*, if G is $K_{2,3}$-free. However, if G is not $K_{2,3}$-free, there is no uniquely determined finest RSP-relation, see Fig. 3.2, 3.3 and 3.4.

By construction, δ_0 places all edges of a $K_{2,3}$-subgraph in the same equivalence class. In many graphs this leads to an RSP-relation which is not finest. On the other hand, the opposite edges of a square that is not contained in a $K_{2,3}$ must always be in the same equivalence class. This motivates us to introduce the following definition.

Definition 3.9 (Relation δ_1)**.** *Two edges $e, f \in E(G)$ are in the* relation δ_1, *$e \; \delta_1 \; f$, if one of the following conditions is satisfied:*

(i) e and f are opposite edges of a square that is not contained in any $K_{2,3}$ subgraph of G.

(ii) e and f are adjacent and there is no square containing e and f, i.e. $(e, f) \in \tau$.

(iii) $e = f$.

If G is $K_{2,3}$-free then it is easy to verify that $\delta_0 = \delta_1$. Proposition 3.8 implies that δ_1^* is contained in any RSP-relation and therefore, that it is a uniquely determined finest RSP-relation on $K_{2,3}$-free graphs. We can summarize this discussion of the properties of finest RSP-relations as follows:

Proposition 3.10. *Let G be an arbitrary graph and R be a finest RSP-relation on $E(G)$. Then it holds that:*

$$\delta_1^* \subseteq R \subseteq \delta_0^*.$$

Moreover, if G is $K_{2,3}$-free, then $\delta_1^* = R = \delta_0^*$.

Basic properties of RSP-relations. In the remainder of this section we collect several basic properties of RSP-relations. Some of these results have originally be obtained for the relation δ in the context of graph products and later were generalized to the unique square property for applications to Cartesian graph bundles. Here we show that the statements remain true for RSP-relations.

The following Lemma was proved in [15] for the relation δ^* and later in [47] assuming the square property. The proof uses only the existence but not the uniqueness of these squares.

However, for completeness, we give the proof here, although it is essentially the same as in [47].

Lemma 3.11. *Let R be an RSP-relation on the edge set of a connected graph G. Then each vertex of G is incident to at least one edge of each R-class.*

Proof. Suppose there exists some class $\varphi_i \sqsubseteq R$ such that there are vertices that are not incident to any φ_i-edge. By connectedness, there must be a pair of adjacent vertices $x, y \in V(G)$ such that x is incident to some φ_i-edge, say $f = [x, w]$, and y is not. The edge $e = [x, y]$ is then in some class $\varphi_k \neq \varphi_i$. By the relaxed square property, e and f span some square $w - x - y - z$ such that $[y, z] \in \varphi_i$, a contradiction. $\square$

Hence, if G is connected and R is an RSP-relation, then $N_\varphi(u) \neq \emptyset$ and $N_{\overline{\varphi}}(u) \neq \emptyset$ for all $u \in V(G)$ and all $\varphi \in R$. Thus, neither G_φ nor $G_{\overline{\varphi}}$ has isolated vertices. Moreover, Lemma 3.11 implies that the number of classes of an RSP-relation cannot exceed $\delta(G)$, the minimal degree of the graph G.

Lemma 3.12. *Let G be a graph and let $\varphi \neq \psi$ be two equivalence classes of an RSP-relation R and let $[v, w] \in \psi$. Then all vertices of G_φ^v are incident to a ψ-edge connecting G_φ^v and G_φ^w. More formally,*

$$N_\psi(v) \cap V(G_\varphi^w) \neq \emptyset \quad \textit{if and only if} \quad N_\psi(x) \cap V(G_\varphi^w) \neq \emptyset$$

holds for all $x \in V(G_\varphi^v)$.

Proof. Consider an arbitrary vertex $x \in V(G_\varphi^v)$. Then there is a path $P := (v = v_0, v_1, \ldots, v_k = x)$ from v to x in G_φ^v. By the relaxed square property, we can construct a walk $Q = (w = w_0, w_1, \ldots, w_k)$ such that $[v_i, w_i] \in \psi$ for all $0 \leq i \leq k$ and $[w_i, w_{i+1}] \in \varphi$ for all $0 \leq i \leq k-1$. Then $w_k \in N_\psi(x)$ and $w_k \in V(G_\varphi^w)$ and therefore, $N_\psi(x) \cap V(G_\varphi^w) \neq \emptyset$. Since $x \in V(G_\varphi^v)$ was arbitrarily chosen, we can conclude that $N_\psi(x) \cap V(G_\varphi^w) \neq \emptyset$ holds for all $x \in V(G_\varphi^v)$. Conversely, if $N_\psi(x) \cap V(G_\varphi^w) \neq \emptyset$ holds for all $x \in V(G_\varphi^v)$, this is trivially fulfilled also for $x = v$. Thus, we have $N_\psi(v) \cap V(G_\varphi^w) = \emptyset$ if and only if $N_\psi(x) \cap V(G_\varphi^w) = \emptyset$ holds for all $x \in V(G_\varphi^v)$. $\square$

The following result was shown in [47] for equivalence relations on the edge set of a graph G that satisfy the square property. Again, the proof uses only the existence but not the uniqueness of these squares nor that they are chordless. For completeness, we give the proof here, although it is essentially the same as in [47].

Lemma 3.13. *Let R be an RSP-relation on $E(G)$ that contains only two equivalence classes φ, $\overline{\varphi}$. Then*

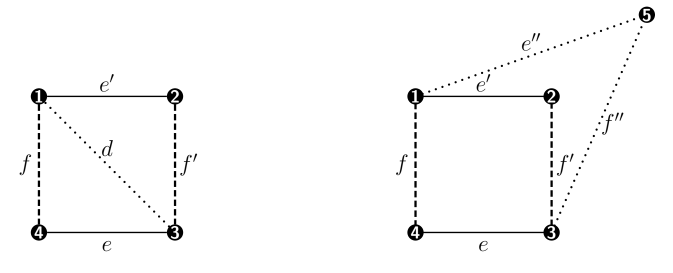
$$|V(G_\varphi^x) \cap V(G_{\overline{\varphi}}^y)| \geq 1$$

for all $x, y \in V(G)$.

Proof. Suppose there are $x, y \in V(G)$ such that $V(G^x_\varphi) \cap V(G^y_{\overline{\varphi}}) = \emptyset$. W.l.o.g. we can assume that G^x_φ and $G^y_{\overline{\varphi}}$ have minimal distance. Let P be a shortest path from G^x_φ to $G^y_{\overline{\varphi}}$. Clearly, the first edge, w.l.o.g. say $[x, w]$, of P must be in $\overline{\varphi}$ and w is not in G^x_φ. Since the distance from $G^w_\varphi \neq G^x_\varphi$ to $G^y_{\overline{\varphi}}$ is smaller than the one from G^x_φ to $G^y_{\overline{\varphi}}$ it follows $V(G^w_\varphi) \cap V(G^y_{\overline{\varphi}}) \neq \emptyset$. Let $v \in V(G^w_\varphi) \cap V(G^y_{\overline{\varphi}})$ and let $Q := (w = w_0, w_1, \dots, w_k = v)$ be a path from w to v in G^w_φ. Since $[x, w] \in \overline{\varphi}$ and $[w, w_1] \in \varphi$, there exists a vertex $x_1 \in V(G)$ such that $[x_1, w_1] \in \overline{\varphi}$ and $[x, x_1] \in \varphi$. Inductively, we obtain a walk $W := (x = x_0, x_1, \dots, x_k)$ such that $[x_i, x_{i+1}] \in \varphi$ for all $0 \leq i \leq k-1$ and $[x_i, w_i] \in \overline{\varphi}$ for all $0 \leq i \leq k$. Thus, $x_k \in V(G^x_\varphi)$ and since $[v, x_k] \in \overline{\varphi}$,we also have $x_k \in V(G^v_{\overline{\varphi}}) = V(G^y_{\overline{\varphi}})$ and therefore $V(G^x_\varphi) \cap V(G^y_{\overline{\varphi}}) \neq \emptyset$. □

If R is convex and has the square property, then $|V(G^x_\varphi) \cap V(G^y_{\overline{\varphi}})| = 1$ holds for all $x, y \in V$ and all $\varphi \sqsubseteq R$ [47]. It is easy to verify, that any convex RSP-relation already has the square property, see Fig. 3.5 and the next explanations: Let $e \in \varphi \sqsubseteq R$ and $f \in \psi \sqsubseteq R$, $\varphi \neq \psi$. e and f span a square consisting of edges e, f, e', f' with $e' \in \varphi$ and $f' \in \psi$. Suppose this square contains a diagonal d (Figure 3.5(a)). If $d \in \varphi$ or $d \in \psi$ then either G_φ or G_ψ is not convex. If $d \notin \varphi \cup \psi$ then $G_{\varphi\cup\psi}$ is not convex. Hence any square spanned by edges from distinct equivalence classes with opposite edges in the same class is chordless if R is convex. Suppose e and f span another square consisting of edges e, f, e'', f'' (Figure 3.5(b)). If $(e'', f'') \in R$ then the subgraph consisting of the edges in the equivalence class containing e'' and f'' is not convex. Hence, let $(e'', f'') \notin R$. We have to consider three cases: either $e'' \notin \varphi \cup \psi$ or $e'' \in \varphi$ or $e'' \in \psi$. If $e'' \notin \varphi \cup \psi$, then $G_{\varphi\cup\psi\cup\chi}$ is not convex, where χ denotes the equivalence class containing f''. Suppose $e'' \in \varphi$. For $[2, 5] \notin E(G)$ or $[2, 5] \notin \varphi$, the subgraph G_φ is not convex. If $[2, 5] \in \varphi$ then $G_{\psi\cup\chi}$ is not convex. Analogously, if $e'' \in \psi$ then G_ψ is not convex for $[3, 5] \notin E(G)$ or $[3, 5] \notin \psi$, and for $[3, 5] \in \psi$, the subgraph $G_{\varphi\cup\chi}$ is not convex. Hence, any square spanned by edges from distinct equivalence classes must be unique if R is convex.

Consequently, Lemma 3.13 applies to RSP-relations as well. In Proposition 5.5 below we will show that the converse is also true.



(a) (b)

Fig. 3.5: RSP-relation R that does not satisfy the square property is not convex: A square $1-2-3-4$ spanned by e and f with $(e, f) \notin R$, **(a)** with diagonal d, **(b)** e and f span another square consisting of edges e, f, e'', f''.

Well-Behaved RSP-Relations. Proposition 3.10 suggests that $K_{2,3}$-subgraphs are to blame for complications in understanding RSP-relations. It will therefore be useful to consider a subclass of RSP-relations that are "well-behaved" on $K_{2,3}$-subgraphs. They will turn out to play a crucial role to establish the connection of RSP-relations, (quasi-)covers, and equitable partitions, see Chapter 4.

Definition 3.14 (Forbidden Coloring, Well-Behaved RSP-Relation). *We fix the notation for the graph $K_{2,3}$ so that $\{x, y\}, \{a, b, c\}$ is the canonical partition of the vertex set. Let R be an equivalence relation on $E(K_{2,3})$ We say that $K_{2,3}$ has a* forbidden coloring *if the edges $[a, x]$, $[x, c]$, and $[y, b]$ are in one equivalence class φ and the other edges are in the union $\overline{\varphi}$ of the classes different from φ.*

An RSP-relation is well-behaved *(on G) if G does not contain a subgraph isomorphic to a $K_{2,3}$ with a forbidden coloring.*

For a graph G and an RSP-relation R consisting of only two equivalence classes we can strengthen this definition. It is easy to verify that in this case the two statements are equivalent:

(i) R is well-behaved

(ii) for each pair of incident edges $[a, b]$, $[a, c]$ which are not in relation R there exists a unique (not necessarily chordless) square $a - b - d - c$ with opposite edges in the same classes, i.e., $([a, b], [c, d]), ([a, c], [b, d]) \in R$.

In the general case (i) implies (ii). To see this, note that if there are incident edges that span more than one square, say SQ_1 and SQ_2, with opposite edges in the same classes, then there is a $K_{2,3}$ with forbidden coloring that consists of the squares SQ_1 and SQ_2. Hence, R cannot be well-behaved. The converse is not true in general, as shown in Fig. 3.1. by the non-well-behaved RSP-relation R' that nevertheless has property (ii).

To obtain well-behaved RSP-relations R on G one can simply use δ_0 and coarsenings of it. That is, *any* equivalence relation R with $\delta_0 \subseteq R$ is well-behaved. In this case, all edges of any $K_{2,3}$-subgraph are in the same equivalence class. However, coarsenings of arbitrary well-behaved RSP-relation R need not be well-behaved, see Fig. 3.1: The well-behaved RSP-relation R on the edge set $E(G)$ of the "diagonalized cube" G has the four equivalence classes $\varphi_1, \varphi_2, \varphi_3$ and φ_4 depicted by solid, zigzag, dotted and dashed edges, respectively. In addition, R satisfies the unique square property. The relation R' with classes φ_3, φ_4 and $\psi_1 = \varphi_1 \cup \varphi_2$, however, is not well-behaved, because the $K_{2,3}$-subgraph with partition $\{1, 6\}$ and $\{2, 4, 5\}$ has a forbidden coloring. Note, R' has the unique square property. Nevertheless, any well-behaved RSP-relation has a well-behaved coarsening with only two equivalence classes:

Lemma 3.15. *Let R be an RSP-relation on the edge set of a graph G. For $\varphi \sqsubseteq R$ denote with R_φ the equivalence relation on $E(G)$ that consists only of the two equivalence classes φ and $\overline{\varphi}$. Then R is well-behaved if and only if R_φ is well behaved for all $\varphi \sqsubseteq R$.*

Furthermore, if R is well-behaved then $R \setminus \{\varphi\}$ is well-behaved.

Proof. Follows immediately from the definition of well-behaved, resp. forbidden coloring. □

Likewise, we can characterize (non-)well-behaved RSP-relations as follows, using squares instead of $K_{2,3}$-subgraphs. If R is not well-behaved, this is equivalent to the existence of squares with two adjacent edges in same class $\varphi \sqsubseteq R$ and others in class(es) different from φ, see Figure 3.6 and the next explanations. It is easy to verify that any $K_{2,3}$(-subgraph) with a forbidden coloring contains such a square. By way of example, consider the square $a-x-c-y$ in Figure 3.6. Conversely, let R be an RSP-relation on $E(G)$ and suppose that G contains a square $a-x-c-y$ with $([a,x],[c,x]) \in R$ and $([a,x],[c,y]),([a,y],[c,x]) \notin R$. By the relaxed square property, $[a,x]$ and $[c,y]$ span a square, say $a-x-b-y$ with opposite edges in the same equivalence class. Hence, there is a complete bipartite graph $K_{2,3}$ with partition $\{x,y\}$ and $\{a,b,c\}$ of $V(K_{2,3})$ and forbidden coloring.

(a) forbidden $K_{2,3}$ (b) forbidden square

Fig. 3.6: Forbidden coloring of a (sub)graph isomorphic to $K_{2,3}$ based on the classes φ and $\overline{\varphi}$ of a (non-well-behaved) RSP-relation. The class $\overline{\varphi}$ might consist of more than one equivalence class. The existence of a forbidden coloring is equivalent to the existence of squares spanned by edges in same equivalence class with opposite edges in different equivalence classes. Such a square contained in the (sub)graph $K_{2,3}$ is shown on the right.

The following result was first proved for δ in [15] and then for equivalence relations with the unique square property in [66]. While it must not hold for RSP-relations in general, it is true for well-behaved RSP-relations and their coarsenings.

Lemma 3.16. *Let R be a (coarsening of a) well-behaved RSP-relation on $E(G)$ and let $[u,v] \in \varphi \sqsubseteq R$. Then R induces a bijection between the ψ-edges incident to u and ψ-edges incident to v for every $\psi \in R$. In particular, the vertices u and v have the same ψ-degree for every $\psi \in R$ with $\psi \neq \varphi$.*

Proof. First, let R be an RSP-relation that is well-behaved and let $[u,v] \in \varphi$. We define a mapping from the ψ-edges incident to u to the ψ-edges incident to v by $[u,x] \mapsto [v,y]$ iff $[u,v]$ and $[u,x]$ span a square $u-v-y-x$ with opposite edges in the same class, i.e., $[u,v],[x,y] \in \varphi$ and $[u,x],[v,y] \in \psi$. Since R is well behaved, the square $u-v-y-x$ is uniquely determined with this property, hence the mapping is well defined. Since $u-v-y-x$ is also the unique square with opposite edges in the same class spanned by the edges $[v,y]$ and $[u,v]$, we conclude the mapping is injective. From the relaxed square property, that is,

Algorithm 1 `Compute RSP-Relation`

INPUT: A connected graph $G = (V, E)$
Compute $R_0 = \delta_1^*$;
$Q \leftarrow \{(e, f) \mid e, f \in E, e \cap f \neq \emptyset\} \setminus R_0$;
$j \leftarrow 0$;
{Note, edges e and f with $(e, f) \in Q$ are adjacent, span a square and are necessarily distinct}
while $Q \neq \emptyset$ **do**
  Take an arbitrary pair $(e, f) \in Q$ with $e \cap f \neq \emptyset$;
  Let $sq_1, \ldots, sq_k$ be the squares spanned by e and f;
  Find the opposite edges e_i of e and f_i of f in sq_i;
  if there is a square sq_i with $(e, e_i) \in R_j^*$ and $(f, f_i) \in R_j^*$ **then**
    $Q \leftarrow Q \setminus \{(e, f), (f, e)\}$;
  else
    take an arbitrary square, say sq_1 {with edge set $E_0 = (e, f, e_1, f_1)$};
    $R_{j+1} \leftarrow R_j^* \cup \{(e, e_1), (e_1, e), (f_1, f), (f, f_1)\}$;
    compute R_{j+1}^*;
    $Q \leftarrow Q \setminus R_{j+1}^*$;
    $j \leftarrow j + 1$;
  end if
end while
$R \leftarrow R_j^*$
OUTPUT: An RSP-relation R on E;

that any ψ-edge incident to u together with $[u, v]$ spans such a square, it follows surjectivity, hence, bijectivity. Since G is simple, it follows that u and v have the same ψ-degree.

Now suppose R is a coarsening of a well-behaved RSP-relation Q. Then each equivalence class $\chi \sqsubseteq R$ is the union of some Q-equivalence classes, $\chi = \bigcup_{\psi \subseteq \chi} \psi$. Since $\psi \cap \psi' = \emptyset$ for any two distinct classes $\psi, \psi' \sqsubseteq Q$ we conclude that the disjoint union of the bijections over the $\psi \subset \chi$ is a bijection between the χ-edges incident to u and the χ-edges incident to v for any $[u, v] \in \varphi \neq \chi$. Clearly, this bijection is again induced by R. It follows immediately that the χ-degrees of u and v are also the same for all $\chi \neq \varphi$. □

Computation. Let us now turn to the computational aspects of RSP-relations. It is an easy task to determine finest relations that have the square property in polynomial time, see [35, 36]. In contrast, it seems to be hard in general to determine one or all finest RSP-relations. We conjecture that the corresponding decision problem is NP- or GI-hard for general graphs. For definitions of NP- and GI-hard see [22, 50].

On the other hand, an efficient polynomial-time solution exists for $K_{2,3}$-free graphs since δ_0 can be constructed efficiently, e.g., by listing all squares [7]. Algorithm 1 serves as a heuristic to find a finest RSP-relation for general graphs. The basic idea is to start from the lower bound $R = \delta_1^*$ and to unite equivalence classes of R stepwisely until an RSP-relation is obtained.

Proposition 3.17. *Let $G = (V, E)$ be a given graph with maximum degree Δ. Algorithm 1 computes an RSP-relation R on E in $O(|V||E|^2\Delta^4)$ time. If G is $K_{2,3}$-free, then Algorithm 1 computes a finest RSP-relation on E.*

Proof. Clearly, δ_1^* must be contained in every RSP-relation R. The set Q contains all adjacent candidate edges (e, f), where we have to ensure that they span a square with opposite edges in the same equivalence class. Since we already computed τ, we can conclude that if e and f are contained in Q, then they span some square. Thus, in Line 10 we check whether there are opposite edges e' of e and f' of f in one of those squares spanned by e end f with $(e, e'), (f, f') \in R_j^*$, i.e., e' and e, resp., f' and f are in the same equivalence class. If so, we can safely remove (e, f) from Q. If not, we will construct a square spanned by e and f with opposite edges in the same class and the pair (e, f) will be removed from Q in the next run of the while-loop (Line 11). To be more precise, we take one of those squares spanned by e and f and add (e, e') and (f, f') to R_j resulting in R_{j+1}. Hence, e and f now span a square with opposite edges in the same class. We then compute the transitive closure R_{j+1}^*. This might result in new pairs $(a, b) \in R_{j+1}^*$ of adjacent edges, which can safely be removed from Q since they are in the same equivalence class, and thus need not span a square with opposite edges in the same class. Hence, we compute $Q \leftarrow Q \setminus R_{j+1}^*$. When Q is empty all adjacent pairs (which span at least one square) are added in a way that at least one square has opposite edges in the same equivalence class. Thus, R satisfies the relaxed square property. Note, if G is $K_{2,3}$-free, then all pairs (e, f) of adjacent edges e and f already span a square with opposite edges in the same class, due to δ_1. Hence, all such pairs (e, f) will be removed from Q, without adding any new pair to R_0^*. In this case we obtain $R = \delta_1^*$.

In order to determine the time complexity we first consider the relation δ_1. Note that there are at most $O(|E|\Delta^2)$ squares in a graph, that can be listed efficiently in $O(|E|\Delta)$ time, see Chiba and Nishizeki [7]. For the computation of δ_1, we first have to check for each square $a - b - c - d$ whether it is contained in a $K_{2,3}$ subgraph or not. Thus, we need to verify whether a and c have a common neighbor $x \notin \{b, d\}$, and, if b and d have a common neighbor $x \notin \{a, c\}$, respectively. If none of the cases occur, i.e., the square is not part of a $K_{2,3}$ subgraph, then we put the pairs $([a, b], [c, d])$ and $([a, d], [b, c])$ to δ_1. This task can be done in $O(\Delta^2)$ time for each square, resulting in an overall time complexity of $O(|E|\Delta^4)$. The relation $\tau \subseteq \delta_1$ can then be computed in $O(|V||E|)$ time [28, Prop. 23.5] and the transitive closure δ_1^* in $O(|E|^2)$ time, [28, Prop. 18.2]. Thus, we end in time complexity $O(|E|^2\Delta^4)$ for the computation of δ_1^*. Finally, we have to check for the at most $|V|\Delta^2$ pairs of adjacent edges whether they already span a square with opposite edges in the same class or not and compute the transitive closure R_{j+1}^* if necessary. Since there are at most $|E|\Delta^2$ squares, $|E| \leq |V|\Delta$, and the transitive closure can be computed in $O(|E|^2)$ time, the latter task can be done in $O(|V||E|^2\Delta^3)$ time. $\square$

As the following example shows, the order in which the squares are examined does matter in the general case, hence Alg. 1 does not produce a finest RSP-relation in general.

Example 3.18. *Consider the complete graph $K_5 = (V, E)$ with vertex set $V = \mathbb{Z}_5$ and natural edge set. After the init step we have $R_0 = \{(e, e) \mid e \in E\}$ and hence, Q contains all pairs of adjacent edges. To obtain a finest RSP-relation, we could start with the pair $([0, 1][1, 4]) \in Q$ that span the square $0-1-4-3$ get as classes $\varphi_1 = \{[0, 1], [3, 4]\}$ and $\varphi_2 = \{[1, 4], [0, 3]\}$ of R_1^*. Continuing with $([0, 1][1, 2]) \in Q$ and the square $0-1-2-3$, we obtain the classes $\varphi_1 \cup \{[2, 3]\}$ and $\varphi_2 \cup \{[1, 2]\}$ of R_2^*. Next, take $([0, 1][0, 4]) \in Q$ and the square $0-1-2-4$, followed by the pair $([0, 1][0, 2]) \in Q$ and the square $0-1-4-2$, resulting in the classes $\varphi_1 = \{[0, 1][2, 3], [3, 4], [2, 4]\}$ and $\varphi_2 = \{[0, 2], [0, 3], [0, 4], [1, 2], [1, 4]\}$ for R_4^*. Finally, take $([0, 1][1, 3]) \in Q$ and the square $0-1-3-4$ to obtain the classes φ_1 and $\varphi_2 \cup \{[1, 3]\}$ for a valid finest RSP-relation, see Lemma 3.20 for further details. Note, the computed RSP-relation is not well-behaved.*

However, if we start with the pair $([0, 1][0, 4]) \in Q$ and square $0-1-3-4$, followed by $([1, 2][1, 3]) \in Q$ and $1-2-4-3$, then $([1, 4][3, 4]) \in Q$ and $1-2-3-4$, next $([0, 1][0, 3]) \in Q$ and $0-1-2-3$ and finally $([0, 2][2, 3]) \in Q$ and $0-2-3-4$, the resulting RSP-relation has only one equivalence class.

Examples: RSP-Relations on Complete and Complete Bipartite Graphs

Since complete graphs and complete bipartite graphs contain large numbers of superimposed $K_{2,3}$ subgraphs they are responsible for much of the difficulties in finding finest RSP-relations. We therefore study their RSP-relations in some detail. Since the proofs are very long, we omit them here for better legibility. They can be found in the appendix.

Lemma 3.19. *Let $V(K_m) = \{0, \ldots, m-1\}$. For $i = 1, \ldots, l := \lfloor \frac{m}{2} \rfloor$ define the set*

$$\varphi_i := \{[x, (x+i) \mod m] \mid x \in \{0, \ldots m-1\}\} \subseteq E(K_m).$$

Then the sets $\varphi_1, \ldots, \varphi_l$ define an RSP-relation R on $E(K_m)$ with equivalence classes $\varphi_1, \ldots, \varphi_l$. If $m \neq 4$, then R is a finest RSP-relation.

Lemma 3.19 implies that the maximum number of classes of a finest RSP-relation is at least $\lfloor \frac{m}{2} \rfloor$. From Lemma 3.5, we infer that the maximum number of classes of a finest RSP-relation on K_m is at most $m-1$, the minimum degree of K_m. In the case of $m = 2^q$, this bound is sharp with the construction in Definition 5.10 and since $K_{2^q} = \boxtimes_{i=1}^q K_2$. However, for all $m \geq 5$ there exists a finest RSP-relation on $E(K_m)$ with only two equivalence classes, as the next lemma shows.

Lemma 3.20. *For $m \geq 5$ and graph K_m, let G_1 be the induced subgraph on vertices $\{0, 1\}$ and G_2 the induced subgraph on $\{2, \ldots, m-1\}$. Then the equivalence relation R with two equivalence classes $\varphi = E(G_1) \cup E(G_2)$ and $\overline{\varphi}$ is a finest RSP relation.*

The relation defined in Lemma 3.20 is also an RSP-relation on K_4. However, it is not the finest RSP-relation in that case.

Example 3.21. *Consider the complete graph $K_9 = K_3 \boxtimes K_3$. Then the construction given in Lemma 3.19 and in Lemma 5.11 define two different RSP-relations $R \ncong S$, for which $K_9/\mathcal{P}^R \cong K_9/\mathcal{P}^S \cong \mathcal{L}K_1$, by Lemma 5.13. Note, R and S have no RSP-relation as common refinement.*

Let us now turn to complete bipartite graphs $K_{m,n}$. W.l.o.g. we may assume that $m \leq n$.

Lemma 3.22. *For $m = n$ let the vertex set of $K_{m,m}$ be given by $V(K_{m,m}) = V(K_2) \times V(K_m)$ and $E(K_{m,m}) = \{[x, y] \mid x, y \in V(K_{m,m})$ s.t. $p_1(x) \neq p_1(y)\}$. Furthermore, let S be an RSP-relation on $E(K_m)$. We define an equivalence relation R on $E(K_{m,m})$ as follows: $(e, f) \in R$ if and only if*

(1) $|p_2(e)| = |p_2(f)| = 1$, or

(2) $|p_2(e)| = |p_2(f)| = 2$ and $(p_2(e), p_2(f)) \in S$.

Then R has the relaxed square property. Moreover, R is a finest RSP-relation on $E(K_{m,m})$ if and only if S is finest RSP-relation on $E(K_m)$.

Lemma 3.23. *For $m < n$ let the vertex set of $K_{m,n}$ be given by $\{x_1, \dots, x_m, y_1, \dots, y_n\}$ such that $E(K_{m,n}) = \{[x_i, y_j] \mid 1 \leq i \leq m, 1 \leq j \leq m\}$. Furthermore, let S be an equivalence relation on the edge set of the induced subgraph $\langle\{x_1, \dots, x_m, y_1, \dots y_m\}\rangle \cong K_{m,m}$ of $K_{m,n}$. We extend S to an equivalence relation R on $E(K_{m,n})$ as follows: For each equivalence class $\varphi' \sqsubseteq S$ we extend φ' to an equivalence class $\varphi \sqsubseteq R$, i.e., we set $\varphi' \subseteq \varphi$ and moreover $[x_j, y_{m+i}]$ is an edge in equivalence class φ if and only if $[x_j, y_{k_i}]$ is an edge in φ' for fixed $k_i \in \{1, \dots, m\}$ for all $i \in \{1, \dots, n - m\}$. Then R has the relaxed square property.*

Obviously, any finer RSP-relation $S' \subset S$ on $E(K_{m,m})$ leads to a finer RSP-relation $R' \subset R$ on $E(K_{m,n})$, constructed from S' as in Lemma 3.23. It is not known yet, if the converse is also true.

The constructions in Lemma 3.22 and Lemma 3.23 together with Lemma 3.19 imply that the maximum number of classes of a finest RSP-relation is at least $\lfloor \frac{m}{2} \rfloor + 1$. From Lemma 3.5, we infer that the maximum number of classes of a finest RSP-relation on $K_{m,n}$ is at most m, the minimum degree of $K_{m,n}$. In the case of $m = 2^q$, this bound is sharp with our considerations for complete graphs K_{2^q} and the constructions in Lemma 3.22 and Lemma 3.23.

Conclusively, we can summarize:

Proposition 3.24. *(1) For all $m > 3$ there exists a nontrivial RSP-relation on $E(K_m)$.*

(2) For all $m, n \geq 2$ there exists a nontrivial RSP-relation on $E(K_{m,n})$.

Chapter 4

Graph Covers, Equitable Partitions and RSP-relations

Graph covers and equitable partitions are intimately related, see [18]. It is known, that if G is a covering graph for a graph H, then G and H must have the same degree refinement matrix [51]. Moreover, Leighton has shown the following:

Theorem 4.1 ([51])**.** *Given any two finite, undirected and connected graphs G and H, the following assertions are equivalent:*

(i) G and H share a common finite cover,

(ii) G and H have the same universal cover,

(iii) G and H share a common (possibly infinite) cover,

(iv) G and H have the same degree refinement matrix.

In Section 4.1, we establish the close connection of covering graphs and (well-behaved) RSP-relations. We will see, i.a., that the layers of a graph w.r.t. an equivalence class of a well-behaved RSP-relation share a common cover, and how such a covering graph can be constructed via this relation. Together with Theorem 4.1, this implies that these layers have a common degree refinement matrix.

In Section 4.2, we will see indeed, how such equitable partitions of the layers arise from well-behaved RSP-relations. Moreover, we show that well-behaved RSP relations on the edge set $E(G)$ also induce equitable partitions of the whole graph G.

4.1 RSP-Relations and Graph Covers

Definition 4.2. *For a graph $G = (V, E)$, an RSP-relation R on E and $\varphi \sqsubseteq R$, let G_φ^x and G_φ^y be two distinct adjacent φ-layers We define the graph $C_{G_\varphi^x, G_\varphi^y}$ in the following way:*

1. *Vertices $V(C_{G_\varphi^x, G_\varphi^y}) = \{[a, b] \in E \mid a \in V(G_\varphi^x), b \in V(G_\varphi^y)\}$ are precisely the edges of G connecting G_φ^x and G_φ^y.*

2. *Two vertices $[a_1, b_1], [a_2, b_2] \in V(C_{G_\varphi^x, G_\varphi^y})$ are adjacent if they are opposite edges of a square $a_1 - b_1 - b_2 - a_2$ in G with $[a_1, a_2] \in E(G_\varphi^x)$ and $[b_1, b_2] \in E(G_\varphi^y)$.*

Lemma 4.3. *Let G be a graph, R an RSP-relation on $E(G)$, and G_φ^x and G_φ^y two distinct adjacent φ-layers for some $\varphi \sqsubseteq R$. Then $C_{G_\varphi^x, G_\varphi^y}$ is a quasi-cover of G_φ^x and G_φ^y. Moreover, if R is well-behaved, then $C_{G_\varphi^x, G_\varphi^y}$ is a cover of G_φ^x and G_φ^y.*

Proof. We define the map $f_1 : V(C_{G_\varphi^x, G_\varphi^y}) \to V(G_\varphi^x)$ by $f_1([a, b]) = a$ where $a \in V(G_\varphi^x)$ and $b \in V(G_\varphi^y)$ and show first that f_1 is a homomorphism, i.e., it maps neighbors in $C_{G_\varphi^x, G_\varphi^y}$ into neighbors in G_φ^x. Let $[a_1, b_1], [a_2, b_2] \in V(C_{G_\varphi^x, G_\varphi^y})$ be adjacent. By construction of edges in $C_{G_\varphi^x, G_\varphi^y}$, there is a square $a_1 - b_1 - b_2 - a_2$ in G with opposite edges $[a_1, b_1]$ and $[a_2, b_2]$. Hence, a_1 and a_2 are adjacent in G_φ^x. Now, let $a = f_1([a, b])$ and $c \in N_{G_\varphi^x}(a)$. Since $[a, c]$ and $[a, b]$ are incident edges of different equivalence classes, they span some square with opposite edges in relation R. Thus there exists a vertex $d \in V(G_\varphi^y)$, such that $[a, b]$ and $[c, d]$ are adjacent in $C_{G_\varphi^x, G_\varphi^y}$ and $f_1([c, d]) = c$. This proves that f_1 is locally surjective and therefore, that $C_{G_\varphi^x, G_\varphi^y}$ is a quasi-cover of G_φ^x.

Let f_1 be defined as above and assume that none of the subgraphs of G that are isomorphic to $K_{2,3}$ have a forbidden coloring. If $f_1([c_1, d_1]) = f_1([c_2, d_2])$ it holds that for $[c_1, d_1], [c_2, d_2] \in N_{C_{G_\varphi^x, G_\varphi^y}}([a, b])$ we have $c_1 = c_2$, by construction of f_1. If $d_1 \neq d_2$, then there is a subgraph of G isomorphic to $K_{2,3}$ with bipartition $\{b, c_1\} \dot\cup \{a, d_1, d_2\}$. Moreover, since $[a, c_1], [b, d_1], [b, d_2] \in \varphi$ and the other edges are, by construction, in $\overline{\varphi}$ we conclude that this subgraph has a forbidden coloring, a contradiction. Thus, $d_1 = d_2$, i.e., the locally surjective map f_1 is also locally injective. Hence, $C_{G_\varphi^x, G_\varphi^y}$ is a cover of G_φ^x.

Arguing analogously for the map $f_2 : V(C_{G_\varphi^x, G_\varphi^y}) \to V(G_\varphi^y)$ with $f_2([a, b]) = b$, $a \in V(G_\varphi^x)$, $b \in V(G_\varphi^y)$, one obtains the desired results for $C_{G_\varphi^x, G_\varphi^y}$ and G_φ^y. □

To illustrate Lemma 4.3 consider the following example: Let $G_1 = C_6$ and $G_2 = C_9$ with vertex sets $\mathbb{Z}_6$ and $\mathbb{Z}_9$ and the canonical edge set definitions. To obtain G add the edges $[k, k \mod 6]$ and $[k, k + 3 \mod 6]$ for $0 \le k \le 8$ connecting G_1 with G_2. Construct an equivalence relation R with two classes $\varphi = E(G_1) \cup E(G_2)$, and $\overline{\varphi}$ comprising the connecting edges. R is a well-behaved RSP-relation on G. It is not hard to verify that C_{G_1, G_2} is a covering graph of C_6 and C_9 and is isomorphic to C_{18}.

For a similar result for the case when G_φ^x and G_φ^y are not distinct, that is $G_\varphi^x = G_\varphi^y$, but there are edges not in φ connecting its vertices, we have to be a bit more careful.

Definition 4.4. *For a graph $G = (V, E)$, an RSP-relation R on E, and $\varphi \sqsubseteq R$, let G^x_φ be some φ-layer. We define the graph $C_{G^x_\varphi, G^x_\varphi}$ in the following way:*

1. *Vertices $V(C_{G^x_\varphi, G^x_\varphi}) = \{(a, b) \mid [a, b] \in E, a, b \in V(G^x_\varphi), [a, b] \in \overline{\varphi}\}$ are edges in $E(G)$ with superimposed orientation (a, b) from a to b, that are not contained in class φ, but that connect vertices of G^x_φ.*

2. *Two directed edges (a_1, b_1) and (a_2, b_2) in $V(C_{G^x_\varphi, G^x_\varphi})$ are adjacent if $[a_1, b_1]$, $[a_2, b_2]$ are opposite edges of a square $a_1 - b_1 - b_2 - a_2$ in G with $[a_1, a_2], [b_1, b_2] \in E(G^x_\varphi)$.*

Remark 4.5. *Since $[a, b] = [b, a]$, it holds that for all edges $[a, b] \in E$, we get two vertices in $V(C_{G^x_\varphi, G^x_\varphi})$ per edge $[a, b] \in E \setminus \varphi$, namely (a, b) and (b, a).*

Lemma 4.6. *For a graph $G = (V, E)$, an RSP-relation R on E, and $\varphi \sqsubseteq R$, let G^x_φ be some φ-layer and assume that there are edges $[a, b] \in E \setminus \varphi$ with $a, b \in V(G^x_\varphi)$. Then $C_{G^x_\varphi, G^x_\varphi}$ is a quasi-cover of G^x_φ with two different locally surjective homomorphisms f_1 and f_2 such that $f_1(h) \neq f_2(h)$ for every $h \in C_{G^x_\varphi, G^x_\varphi}$. Moreover, if R is well-behaved, then $C_{G^x_\varphi, G^x_\varphi}$ is twice a cover of G^x_φ, i.e., there are at least two different covering maps.*

Proof. Proof is the same as for Lemma 4.3 by defining $f_1((a, b)) = a$ and $f_2((a, b)) = b$. □

If every vertex of G^x_φ is incident with exactly one edge that is not in φ but connects two vertices of G^x_φ, then $G^x_\varphi \cong C_{G^x_\varphi, G^x_\varphi}$ and the edges in $\overline{\varphi}$ induce an automorphism of G^x_φ without fixed vertices by setting $f(a) = b$ whenever $[a, b] \in \overline{\varphi}$.

As an example consider the graph G with $V(G) = \mathbb{Z}_6$ and $E(G) = \varphi \dot{\cup} \overline{\varphi}$ such that $\varphi = \{[k, k+1 \mod 6] \mid 0 \leq k \leq 5\}$, i.e., $G_\varphi \cong C_6$ and $\overline{\varphi} = \{[1, 4], [2, 5], [3, 6]\}$. We then have $V(C_{G^x_\varphi, G^x_\varphi}) = \{(0, 3), (1, 4), (2, 5), (3, 0), (4, 1), (5, 2)\}$ and $C_{G^x_\varphi, G^x_\varphi}$ has edges $E(C_{G^x_\varphi, G^x_\varphi}) = \{[(0, 3), (1, 4)], [(1, 4), (2, 5)], [(2, 5), (3, 0)], [(3, 0), (4, 1)], [(4, 1), (5, 2)], [(5, 2), (0, 2)]\}$, that is $C_{G^x_\varphi, G^x_\varphi} \cong C_6 \cong G_\varphi$. The induced automorphism is given by $f(k) = k+3 \mod 6$, $k = 0, \ldots, 5$.

Lemma 4.3 and Lemma 4.6 together highlight a connection between graph bundles and graphs with relaxed square property. For an RSP-relation R on G we see that the connected components G_φ correspond to fibers, while the graph $G_{\overline{\varphi}}/\mathcal{P}^R_\varphi$ has the role of the base graph. Such decomposition is a graph bundle if and only if edges connecting G^x_φ and G^y_φ for arbitrary x, y induce an isomorphism. In our language, this is equivalent to the condition $C_{G^x_\varphi, G^y_\varphi} \cong G^x_\varphi \cong G^y_\varphi$ for arbitrary x, y, provided that G^x_φ and G^y_φ are connected by an edge. Graphs with a nontrivial RSP-relation are therefore a natural generalization of graph bundles.

Another interesting question is how two graphs G_1 and G_2 can be connected by additional edges so that $\varphi = E(G_1) \cup E(G_2)$ and $\overline{\varphi}$ comprises the connecting edges and $R = \{\varphi, \overline{\varphi}\}$ is an RSP-relation.

Lemma 4.7. *Let G_1, G_2, and G be graphs and $f_1 : G \to G_1$, $f_2 : G \to G_2$ be locally surjective homomorphisms. Then there exists a graph $H = (V, E)$ and an RSP-relation R on E with equivalence classes φ, $\overline{\varphi}$ such that*

(1) $V = V(G_1) \cup V(G_2)$*, and*

(2) $\varphi = E(G_1) \cup E(G_2)$*.*

Note, it is allowed to have $G_1 = G_2$*. In this case,* H *might have loops and double edges.*

Proof. For given graphs G_1, G_2, G and locally surjective homomorphisms $f_i : G \to G_i$, $i = 1, 2$ construct the graph H as follows: For $x \in V(G_1)$ and $y \in V(G_2)$ add an edge $[x, y]$ if and only if there exists $g \in V(G)$ such that $f_1(g) = x$ and $f_2(g) = y$. We set $[x, y] \in \overline{\varphi}$. It is clear, that R is an equivalence relation. We have to show, that R is an RSP-relation. Let $[x_1, x_2] \in E(G_1)$ and $[x_1, y_1]$ be an added edge. Then there exists $g_1 \in V(G)$, such that $f_1(g_1) = x_1$ and $f_2(g_1) = y_1$. Since f_1 is a locally surjective homomorphism, there exists a vertex g_2 as a neighbor of g_1, such that $f_1(g_2) = x_2$. Let $y_2 = f_2(g_2)$. Then y_2 and x_2 are connected by an added edge and y_1, y_2 are adjacent since f_2 is a homomorphism. Thus $[x_1, x_2]$ and $[x_1, y_1]$ lie on a square with opposite edges in relation R.

If $G_1 = G_2$, then just identify vertices of two copies of G_1. □

Lemma 4.8. *Let* G *and* G' *be two graphs. Then there exists a graph* $H = (V, E)$ *and a well-behaved RSP-relation* R *with two equivalence classes* φ, $\overline{\varphi}$ *such that*

(1) $V = V(G) \cup V(G')$*, and*

(2) $\varphi = E(G) \cup E(G')$*, and*

(3) each vertex of $V(G)$ *is incident to exactly one* $\overline{\varphi}$*-edge*

if and only if G *is a cover of* G'*.*

Proof. Let $H = (V, E)$ be a graph with well-behaved RSP-relation R on E as claimed. Then, we can consider G, G' as φ-layers. By Lemma 4.3, $C_{G',G}$ is a cover of G' and G. Since each vertex in $V(G)$ is incident with exactly one $\overline{\varphi}$-edge, we see that for covering map $f_1 : C_{G',G} \to G$ holds $|f_1^{-1}(u)| = 1$ for all $u \in H$ which implies f_1 is also injective, thus an isomorphism.

For the converse, assume G is a cover of G'. Then G is a cover of G and G' and thus G and G' can be connected as in the prove of Lemma 4.7. Since clearly $G \cong G$ and thus the covering map $p : G \to G$ is in particular injective, each vertex is, by construction, incident to exactly one $\overline{\varphi}$-edge. This in turn implies, H contains no square $w - x - y - z$ such that $z \in V(G)$ and $[w, z], [y, z] \in \overline{\varphi}$. On the other hand, there is no square $w - x - y - z$ contained in H with $[w, x], [x, y] \in E(G) \subseteq \varphi$ and $[w, z], [y, z] \in \overline{\varphi}$, i.e., $z \in V(G')$, since otherwise the restriction of the covering map $p' : G \to G'$ to $N_G(x)$ (w.l.o.g. we can assume p to be the identity mapping) would not be injective, a contradiction. Hence, we can conclude that R is well-behaved. □

Notice that checking if H is a cover of G is in general NP-hard [1]. Therefore, also connecting two graphs as described in the proof of Lemma 4.8 is NP-hard. On the other hand, one can connect two arbitrary graphs G_1, G_2 such that all vertices of G_1 are linked to all vertices of G_2. Then, the relation defined by the classes $\varphi = E(G_1) \cup E(G_2)$ and $\overline{\varphi}$ that consists of all added edges between G_1 and G_2 is an RSP-relation. This implies that any two graphs have a common finite quasi-cover. However, this is not true for covers, just take K_2 and K_3 as an example.

For a given graph G and an RSP-relation R, one can consider the subgraph G_φ, $\varphi \sqsubseteq R$ as one layer and all other edges of G not contained in G_φ as connecting edges. Notice, connectivity is not explicitly needed in Definition 4.4 and Lemma 4.6, and thus, they can be extended to C_{G_φ, G_φ}. Moreover, any spanning subgraph H of a graph G induces an equivalence relation R with two equivalence classes $E(H)$ and $E(G) \setminus E(H)$. Hence, $C_{H,H}$ is well defined and thus, Lemma 4.6 and 4.7 imply the following result.

Theorem 4.9. *A graph G has an RSP-relation with two equivalence classes if and only if there exists a (possibly disconnected) spanning subgraph $H \subsetneq G$ and $C_{H,H}$ is a quasi-cover of H.*

On the set of graphs $\mathfrak{G}$ we consider the relation $G_1 \sim G_2$ if G_1 and G_2 have a common finite cover.

Proposition 4.10. *The relation $\sim$ on $\mathfrak{G}$ is an equivalence relation.*

Proof. Relation $\sim$ is clearly reflexive and symmetric. By assumption, the graphs G_1 and G_2 have a common cover H_{12}, and G_2 and G_3 have a common cover H_{23}. By Lemma 4.8, H_{12} and G_2 as well as H_{23} and G_2 can be connected without forbidden colorings of $K_{2,3}$. Let E be the set of all edges connecting G_2 and H_{12} and E' the set of edges connecting G_2 and H_{23}. Since every cover of H_{12} and H_{23} is a cover of G_1, G_2 and G_3, it is sufficient to find a cover of H_{12} and H_{23}. Therefore, it suffices to connect H_{12} and H_{23} without forbidden colorings of $K_{2,3}$. Define edges connecting H_{12} and H_{23} by connecting $h \in V(H_{12})$ and $h' \in V(H_{23})$ if there exists a vertex $v \in V(G_2)$ such that $[h, v] \in E$ and $[v, h'] \in E'$.

First we check that $E(H_{12}) \cup E(H_{23})$ and the connecting edges form two equivalence classes of an RSP relation. W.l.o.g. assume $[h_1, h_2] \in E(H_{12})$ and $[h_1, h'_1]$, $h'_1 \in V(H_{23})$ is a connecting edge. Then there exists $v_1 \in V(G_2)$ such that $[h_1, v_1] \in E$ and $[v_1, h'_1] \in E'$. Since edges E are defined by a local bijection between H_{12} and G_2, there exist $v_2 \in V(G_2)$, a neighbor of v_1, such that $[h_2, v_2] \in E$. Similarly, since E' is defined by a local bijection between H_{23} and G_2, there exists $h'_2 \in V(H_{23})$, a neighbor of h'_1, such that $[v_2, h'_2] \in E'$. Therefore there exists a square $h_1 - h'_1 - h'_2 - h_2$ with $[h_1, h_2], [h'_1, h'_2] \in E(H_{12}) \cup E(H_{23})$ and $[h_1, h_2], [h'_1, h'_2]$ being connecting edges. This proves that the relation R with equivalence classes $E(H_{12}) \cup E(H_{23})$ and the set of connecting edges is an RSP relation.

It remains to prove that R is well-behaved. By symmetry, it is enough to prove that there exists no vertices $h_1, h_2, h_3 \in V(H_{12})$ and $h'_1, h'_2 \in V(H_{23})$ with $[h_1, h_2], [h_1, h_3] \in E(H_{12})$, $[h'_1, h'_2] \in E(H_{23})$ and added edges $[h_1, h'_1], [h_2, h'_2]$ and $[h_3, h'_2]$. For the sake of contradiction, suppose such vertices exist. By the construction of the added edges, there exist vertices

$v_1, v_2, v_3 \in V(G_2)$ such that $[h_1, v_1], [h_2, v_2], [h_3, v_3] \in E$ and $[v_1, h'_1]$, $[v_2, h'_2]$, $[v_3, h'_2] \in E'$. Since edges in E are obtained from a covering map of H_{12} to G_2 we see that v_1, v_2 and v_3 are distinct vertices. But also the edges in E' are obtained from a covering map of H_{23} to G_2 therefore $[v_2, h'_2] = [v_3, h'_2]$ and thus $v_2 = v_3$, a contradiction. □

We have proven Proposition 4.10 here by elementary means to keep this presentation self-contained. It also follows from Theorem 4.1. Moreover, the relation $\sim$ is just the symmetric and transitive closure of the partial order $\leq_B$, defined on the set of graphs $\mathfrak{G}$ as follows: $G_1 \leq_B G_2$ if G_1 is a cover of G_2 [18][1].

Corollary 4.11. *Let G be a connected graph and let R be a well-behaved RSP-relation on $E(G)$. Then there exists a common covering graph for all φ-layers G_φ^x for each equivalence class $\varphi \sqsubseteq R$.*

Proof. This result is an immediate consequence of the connectedness of G, Lemma 4.3 and Proposition 4.10. □

The latter corollary can also be extended to coarsenings of well-behaved RSP-relations.

Corollary 4.12. *Let G be a connected graph and let S be a coarsening of a well-behaved RSP-relation R on $E(G)$. Then there exists a common covering graph for all φ-layers G_φ^x for each equivalence class $\varphi \sqsubseteq S$.*

Proof. Let $\varphi \sqsubseteq S$. Since R is a refinement of S, we have $\varphi = \bigcup_{\chi \subseteq \varphi} \chi \sqsubseteq R$. Moreover, let $\psi \sqsubseteq R$ such that $\psi \cap \chi = \emptyset$. By Lemma 3.15, it follows that the equivalence relation $Q = \{\varphi, \psi\}$ is a well-behaved RSP-relation on the edge set of the subgraph $(V(G), \varphi \cup \psi)$ of G. Now applying Corollary 4.11, the assertion follows. □

In terms of Leighton's theorem, the corollary could be read in the following way: For a graph G with a well-behaved RSP-relation on $E(G)$ and some fixed equivalence class φ all the graphs $\{G_\varphi^x\}$ have the same universal cover.

Under certain conditions it is possible to refine a given RSP-relation.

Lemma 4.13. *Let $G = (V, E)$ be a connected graph and R a well-behaved RSP-relation on E. Assume that for one equivalence class $\varphi \sqsubseteq R$ the graph G_φ has two connected components G_φ^x and G_φ^y. The next two statements are equivalent:*

1. *There is a well-behaved refined RSP-relation $R' \subsetneq R$ such that $\varphi = \chi_1 \cup \chi_2$ with $\chi_1, \chi_2 \sqsubseteq R'$*

2. *$C_{G_\varphi^x, G_\varphi^y}$ has a non-trivial RSP-relation Q such that $(e, f) \in Q$ iff $(e', f') \in R'$ for all $e, f \in p_1^{-1}(e') \cup p_1^{-1}(f') \cup p_2^{-1}(e') \cup p_2^{-1}(f')$ and for all $e', f' \in E(G_\varphi^x) \cup E(G_\varphi^y)$, where $p_1 : C_{G_\varphi^x, G_\varphi^y} \to G_\varphi^x$, resp., $p_2 : C_{G_\varphi^x, G_\varphi^y} \to G_\varphi^y$.*

[1] In fact, this assertion was only shown for connected graphs. However, the authors stated that the results of their contribution can be straightforwardly generalized to disconnected graphs.

In other words, R can be refined to R' if and only if edges of G_φ^x, resp., G_φ^y that map on the same edges via the covering projection are in the same class w.r.t. Q.

Proof. If there is a finer RSP-relation R', every square $a_1 - b_1 - b_2 - a_2$ with $a_1, a_2 \in V(G_\varphi^x)$ and $b_1, b_2 \in V(G_\varphi^y)$ has edges $[a_1, a_2]$ and $[b_1, b_2]$ in the same class by the relaxed square property and since R is well-behaved. Thus, an equivalence relation on $E(G_\varphi^x)$ and $E(G_\varphi^y)$ can be lifted to an equivalence relation on $E(C_{G_\varphi^x, G_\varphi^y})$ in a natural way. One can check that it has the relaxed square property by using that the respective relations on $E(G_\varphi^x)$ and $E(G_\varphi^y)$ have the relaxed square property.

Conversely, we define a finer RSP-relation on $E(G_\varphi^x)$ and $E(G_\varphi^y)$ from the RSP-relation on $E(C_{G_\varphi^x, G_\varphi^y})$ by setting $(e', f') \in R'$ iff $(e, f) \in Q$ for some $e \in p_1^{-1}(e'), f \in p_1^{-1}(e')$. □

Let R be a well-behaved RSP-relation on G, e.g., $R = \delta_0$, and suppose there is a finer RSP-relation R' in which an equivalence class φ is split into two equivalence classes φ_1 and φ_2. Let $\{G_\varphi^{x_i}\}$ be the connected components of G_φ. Then φ_1 and φ_2 induce an RSP-relation on each $G_\varphi^{x_i}$. Consider two components $G_\varphi^{x_1}$ and $G_\varphi^{x_2}$ that are connected by some edges (in other classes). From the proof of Lemma 4.13 we observe that an RSP-relation on $E(G_\varphi^{x_1})$ already defines an RSP-relation on $C_{G_\varphi^{x_1}, G_\varphi^{x_2}}$, which in turn defines an RSP-relation on $G_\varphi^{x_2}$ and thus on all φ-layers $G_\varphi^{x_i}$. If multiple splits of φ exist, they are fixed by choosing one on any $G_\varphi^{x_i}$.

Now consider the graph G consisting of two copies of $K_{2,3}$ and all edges connecting them and the equivalence relation whose two classes are the edges of the two copies of $K_{2,3}$ and the connecting edges, respectively. The discussion above implies that we can split the first class independently on the two copies of $K_{2,3}$. Thus, we cannot generalize the result above to RSP-relations with forbidden colorings.

4.2 RSP-Relations and Equitable Partitions

As illustrated at the beginning of this chapter, Graph covers and equitable partitions are intimately connected. The following result is an immediate consequence of Lemmas 4.3 and 4.6 from the previous section and sharpens the assertion in Lemma 3.12 for well-behaved RSP-relations and their coarse grainings.

Lemma 4.14. *Let G be a graph and let R be a (coarsening of a) well-behaved RSP-relation. Furthermore, let $\varphi, \psi \sqsubseteq R$ with $\varphi \neq \psi$ and $v, w \in V(G)$. Then all vertices of G_φ^v have the same number of incident ψ-edges connecting G_φ^v and G_φ^w. More formally,*

$$|N_\psi(v) \cap V(G_\varphi^w)| = |N_\psi(x) \cap V(G_\varphi^w)|$$

holds for all $x \in V(G_\varphi^v)$.

Proof. Let $Q \subseteq R$ be a well-behaved RSP-relation. Then ψ is the disjoint union of some equivalence classes $\chi \sqsubseteq Q$, $\psi = \bigcup_{\chi\subseteq\psi} \chi$. Thus, we have

$$|N_\psi(x) \cap V(G^w_\varphi)| = \sum_{\chi\subseteq\psi} |N_\chi(x) \cap V(G^w_\varphi)|.$$

Therefore, it suffices to show that $|N_\chi(v) \cap V(G^w_\varphi)| = |N_\chi(x) \cap V(G^w_\varphi)|$ holds for all $x \in V(G^v_\varphi)$ and all $\chi \sqsubseteq Q$ with $\chi \subseteq \psi$. Note, it holds $\chi \cap \varphi = \emptyset$ for all $\chi \subseteq \psi$.

By Lemma 3.6 the relation R' consisting of classes φ, χ on the edge set of the spanning subgraph $H = (V(G), \varphi\cup\chi)$ of G is also an RSP-relation. Applying Lemma 3.15, we conclude that R' is well-behaved. Moreover, since $H_\varphi = G_\varphi$ and $N_{H_\chi}(x) = N_{G_\chi}(x) = N_\chi(x)$ for all $x \in V(G) = V(H)$, and $\chi = \overline{\varphi}$ w.r.t. H, it suffices to prove $|N_{H_{\overline{\varphi}}}(v) \cap V(H^w_\varphi)| = |N_{H_{\overline{\varphi}}}(x) \cap V(H^w_\varphi)|$ holds for all $x \in V(H^v_\varphi)$.

If there is no edge in $H_{\overline{\varphi}}$ connecting H^v_φ and H^w_φ the assertion is clearly true. Therefore assume now that they are connected by an edge. By Lemmas 4.3 and 4.6, $C_{H^v_\varphi, H^w_\varphi}$ is a cover of H^v_φ with covering map f_1 as defined in Lemmas 4.3 resp. 4.6. By definition of f_1, it holds $|f_1^{-1}(x)| = |N_{H_{\overline{\varphi}}}(x) \cap V(H^w_\varphi)|$, which is the same for all $x \in V(H^v_\varphi)$. □

Corollary 4.15. *Let G be a connected graph and R be a (coarsening of a) well-behaved RSP-relation on $E(G)$. Then $P^R_{\overline{\varphi}} = \left\{V(G^x_{\overline{\varphi}}) \mid x \in V(G)\right\}$ is an equitable partition of the graph G_φ for every equivalence class φ of R.*

Proof. This follows immediately from $V(G) = V(G_\varphi)$, the fact that $P^R_{\overline{\varphi}}$ is a partition of $V(G)$, and Lemma 4.14. □

If G_φ is an induced subgraph of G we have $G_\varphi/\mathcal{P}^R_{\overline{\varphi}} \cong \mathcal{N}(G/\mathcal{P}^R_{\overline{\varphi}})$ which follows from the fact that $[G^x_{\overline{\varphi}}, G^y_{\overline{\varphi}}]$ is an edge in $\mathcal{N}(G/\mathcal{P}^R_{\overline{\varphi}})$ if and only if there is an edge in G connecting a vertex in $V(G^x_{\overline{\varphi}})$ with a vertex in $V(G^y_{\overline{\varphi}})$. This edge must be in φ, since otherwise $G^x_{\overline{\varphi}} = G^y_{\overline{\varphi}}$, and hence it is in G_φ.

Remark 4.16. *The quotient graphs $B_{\overline{\varphi}} := G_\varphi/\mathcal{P}^R_{\overline{\varphi}}$ provide a direct connection to the theory of graph bundles since $B_{\overline{\varphi}}$ coincides with the base graph of the bundle presentation $(G, p_{\overline{\varphi}}, B_{\overline{\varphi}})$ of G provided $\overline{\varphi}$ is 2-convex and R has the unique square property [46, 57]. Moreover, it can easily be shown that G has a graph bundle presentation $(G, p, G_\varphi/\mathcal{P}^R_{\overline{\varphi}})$ over a simple base if and only if $p : G_\varphi \to G_\varphi/\mathcal{P}^R_{\overline{\varphi}}$ is a covering projection [18, 46, 57].*

Theorem 4.17. *Let R be a (coarsening of a) well-behaved RSP-relation on the edge set $E(G)$ of a connected graph G. Then $\mathcal{P}^R = \{V_R(x) \mid x \in V(G)\} = \left\{\bigcap_{\varphi\sqsubseteq R} V(G_{\overline{\varphi}}(x)) \mid x \in V(G)\right\}$ is an equitable partition of G.*

To prove the Theorem, we first show the following:

Lemma 4.18. *Let G be a connected graph and R be a (coarsening of a) well-behaved RSP-relation on $E(G)$. Then for an arbitrary equivalence class φ of R holds:*

(1) $N_\varphi(x) \cap V_R(y) \neq \emptyset$ if and only if $N_\varphi(u) \cap V_R(y) \neq \emptyset$ for all $u \in V_R(x)$.

(2) $N_\varphi(x) \cap V_R(y) \neq \emptyset$ *implies* $N_\varphi(x) \cap V_R(y) = N_\varphi(x) \cap V(G^y_\varphi)$.

Proof. (1) Let $N_\varphi(x) \cap V_R(y) \neq \emptyset$ and hence, $N_\varphi(x) \cap V(G^y_\psi) \neq \emptyset$ for all $\psi \sqsubseteq R$. Thus, there exists a vertex $z \in V(G)$ with $[x,z] \in \varphi$ such that $z \in V(G^y_\psi)$ for all $\psi \sqsubseteq R$. Note, $z \in V(G^x_\psi)$ because for all $\varphi \neq \psi$ there is a path that is not in ψ which is the particular edge $[x,z] \in \varphi$. Therefore, $G^x_\psi = G^y_\psi$ for all $\psi \neq \varphi$.

Now let $u \in V_R(x)$. Hence $u \in V(G^x_\psi) = V(G^y_\psi)$ for all $\psi \neq \varphi$. From Lemma 4.14 and the fact that $N_\varphi(x) \cap V(G^y_\varphi) \neq \emptyset$, we can conclude that $N_\varphi(u) \cap V(G^y_\varphi) \neq \emptyset$, i.e., there exists a vertex $w \in V(G^y_\varphi)$ such that $[u,w] \in \varphi$. This implies $w \in V(G^u_\psi) = V(G^y_\psi)$ for all $\psi \neq \varphi$ and therefore $w \in V_R(y)$, hence $N_\varphi(u) \cap V_R(y) \neq \emptyset$. Conversely, if $N_\varphi(u) \cap V_R(y) \neq \emptyset$ for all $u \in V_R(x)$, this is trivially fulfilled for $u = x$.

(2) Let $z \in N_\varphi(x) \cap V_R(y)$, that is, $z \in N_\varphi(x)$ and $z \in V(G^y_\psi)$ for all $\psi \sqsubseteq R$, in particular, $z \in V(G^y_\varphi)$. Hence, $z \in N_\varphi(x) \cap V(G^y_\varphi)$ and therefore we have $N_\varphi(x) \cap V_R(y) \subseteq N_\varphi(x) \cap V(G^y_\varphi)$. Now, let $z \in N_\varphi(x) \cap V(G^y_\varphi)$, which is equivalent to $[x,z] \in \varphi$ and $z \in V(G^y_\varphi)$. It follows $z \in V(G^x_\psi)$ for all $\psi \neq \varphi$ and thus $z \in V(G^y_\psi)$ for all $\psi \sqsubseteq R$ since $N_\varphi(x) \cap V_R(y) \neq \emptyset$. Hence, $z \in N_\varphi(x) \cap V_R(y)$ and therefore $N_\varphi(x) \cap V(G^y_\varphi) \subseteq N_\varphi(x) \cap V_R(y)$, from which we can conclude equality of the sets. □

Proof of Theorem 4.17. By construction, $\mathcal{P}^R$ is a partition of $V(G)$. It remains to show that this partition is equitable, that is, we have to show that for arbitrary $u, x, y \in V(G)$ with $u \in V_R(x)$ holds

$$|N_G(u) \cap V_R(y)| = |N_G(x) \cap V_R(y)|. \tag{4.1}$$

Notice that for arbitrary $x \in V(G)$ we have $N_G(x) = \bigcup_{\varphi \sqsubseteq R} N_\varphi(x)$ and $N_\varphi(x) \cap N_\psi(x) = \emptyset$ for $\varphi \neq \psi$. Hence we have $|N_G(x) \cap V_R(y)| = \sum_{\varphi \sqsubseteq R} |N_\varphi(x) \cap V_R(y)|$. Therefore, it suffices to show

$$|N_\varphi(u) \cap V_R(y)| = |N_\varphi(x) \cap V_R(y)| \quad \forall \varphi \sqsubseteq R$$

to prove Equation (4.1). This equality, however, follows immediately from Lemma 4.18 together with Lemma 4.14. □

Chapter 5

RSP-Relations and Product Structures

RSP-relations generalize the product relation σ in a way that some but not all properties of the layers are retained. In this manner, we can view graphs, that admit nontrivial RSP-relations on their edge sets to have *relaxed product structure*. In fact, an RSP-relation R on the edge set $E(G)$ of a graph G induces a partition on the vertex set whose quotient graphs exhibit a natural, rich product structure, even when G itself is prime. This will be shown in the first section of this chapter, where we will also examine the behavior of these quotient graphs w.r.t. refinements and coarse grainings of the particular relation R.

In Section 5.2, we will see for each of the products $\circledast \in \{\Box, \boxtimes, \times\}$, how RSP-relations of the factors will be transferred to the product graph. Moreover, we will explore in what manner the structure of the quotient of a product graph, that was found in Section 5.1, depends on the structure of the quotient graphs of its factors, and the specific product.

5.1 Product Structures of Quotient Graphs

The connected components of a given equivalence class of the product relation σ, i.e., the fibers of G w.r.t. a given factor F, form a natural partition $\mathcal{P}_F$ of the vertex set of G. It is well known (see e.g. [28]) that G then has a representation as $G \cong (G/\mathcal{P}_F)\Box F$. The following main result of this section generalizes this observation to RSP-relations.

Theorem 5.1. *Let R be an RSP-relation on the edge set $E(G)$ of a connected graph G. Then*

$$G/\mathcal{P}^R \cong \mathop{\Box}_{\varphi \sqsubseteq R} G_\varphi/\mathcal{P}^R_{\varphi}.$$

Proof. Let $\varphi_1, \dots, \varphi_n$ denote the equivalence classes of R. Let $x, v_1, \dots, v_n \in V(G)$, where the v_i need not necessarily be distinct. If $x \in V(G^{v_i}_{\varphi_i})$ for all $i = 1, \dots, n$ then $V_R(x) = \bigcap_{i=1}^n V(G^{v_i}_{\varphi_i})$.

Remark, that for $1 \leq i \leq n$ the vertex set of $G_{\varphi_i}/\mathcal{P}^R_{\overline{\varphi}_i}$ is given by $V(G_{\varphi_i}/\mathcal{P}^R_{\overline{\varphi}_i}) = \{G^{v_i}_{\overline{\varphi}_i} \mid v_i \in V(G)\}$. Hence, we have

$$V(\mathop{\Box}_{i=1}^{n} G_{\varphi_i}/\mathcal{P}^R_{\overline{\varphi}_i}) = \left\{(G^{v_1}_{\overline{\varphi}_1}, \ldots, G^{v_n}_{\overline{\varphi}_n}) \mid v_i \in V(G), i = 1, \ldots, n\right\},$$

where $(G^{v_1}_{\overline{\varphi}_1}, \ldots, G^{v_n}_{\overline{\varphi}_n}) = (G^{u_1}_{\overline{\varphi}_1}, \ldots, G^{u_n}_{\overline{\varphi}_n})$ if and only if $u_i \in V(G^{v_i}_{\overline{\varphi}_i})$ for all $i = 1, \ldots, n$.

We define a mapping $V(G/\mathcal{P}^R) \to V(\Box_{i=1}^n G_{\varphi_i}/\mathcal{P}^R_{\overline{\varphi}_i})$ as follows:

$$V_R(x) \mapsto (G^{v_1}_{\overline{\varphi}_1}, \ldots, G^{v_n}_{\overline{\varphi}_n})$$

iff $x \in V(G^{v_i}_{\overline{\varphi}_i})$ for all $i = 1, \ldots, n$.

For all $x \in V(G)$ there exist v_i, $i = 1, \ldots, n$ such that $x \in V(G^{v_i}_{\overline{\varphi}_i})$, e.g. choose $v_i = x$. And since from $x \in V(G^{v_i}_{\overline{\varphi}_i})$ and $x \in V(G^{u_i}_{\overline{\varphi}_i})$ follows $G^{v_i}_{\overline{\varphi}_i} = G^{u_i}_{\overline{\varphi}_i}$, this mapping is well defined.

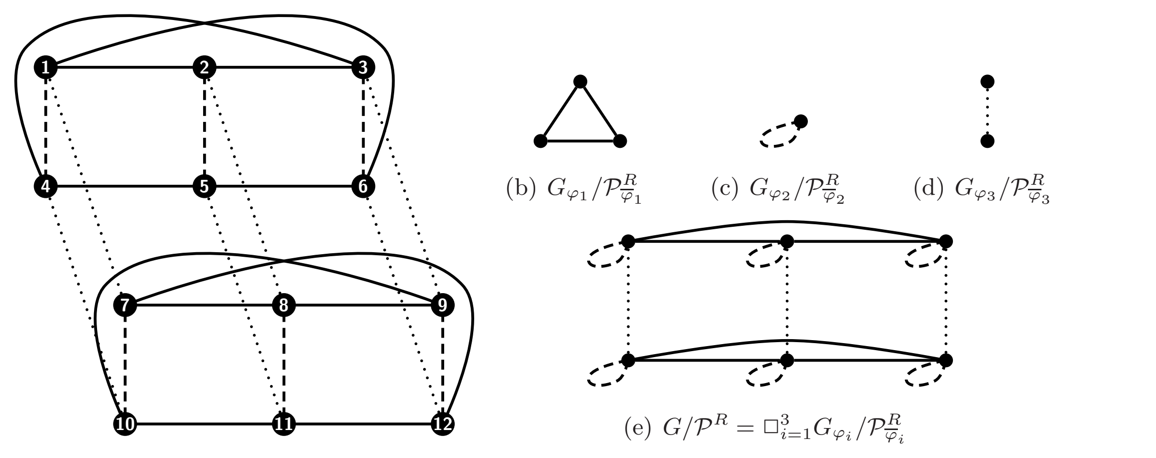
Due to the fact that $x \in V(G^{v_i}_{\overline{\varphi}_i})$ and $y \in V(G^{v_i}_{\overline{\varphi}_i})$ implies $G^{x}_{\overline{\varphi}_i} = G^{y}_{\overline{\varphi}_i}$, we can conclude that this mapping is injective. To prove surjectivity, it suffices to show, that $\cap_{i=1}^n V(G^{v_i}_{\overline{\varphi}_i}) \neq \emptyset$ for arbitrary $v_i \in V(G)$. We show by induction for all $k \leq n$ holds $\cap_{i=1}^k V(G^{v_i}_{\overline{\varphi}_i}) \neq \emptyset$. For $k = 1$ this is trivially fulfilled. Let $k \geq 1$ and assume $\cap_{i=1}^k V(G^{v_i}_{\overline{\varphi}_i}) \neq \emptyset$. We have to show, that this implies $\cap_{i=1}^{k+1} V(G^{v_i}_{\overline{\varphi}_i}) \neq \emptyset$. From the induction hypothesis, we can conclude there must be a vertex $x \in V(G)$ such that $x \in V(G^{v_i}_{\overline{\varphi}_i})$ for all $i = 1, \ldots, k$ and hence $\cap_{i=1}^k V(G^{v_i}_{\overline{\varphi}_i}) = \cap_{i=1}^k V(G^{x}_{\overline{\varphi}_i})$ for all $i = 1, \ldots, k$. Therefore, we have to show

$$V(G^x_{\varphi_{k+1}}) \subseteq \bigcap_{i=1}^{k} V(G^x_{\overline{\varphi}_i}). \tag{5.1}$$

From that and Lemma 3.13 we can conclude $\emptyset \neq V(G^x_{\varphi_{k+1}}) \cap V(G^{v_{k+1}}_{\overline{\varphi}_{k+1}}) \subseteq \bigcap_{i=1}^{k+1} V(G^{v_i}_{\overline{\varphi}_i})$, from what the assumption follows.

Let $y \in V(G^x_{\varphi_{k+1}})$. Then there exists a path Q from x to y such that all edges of Q are in class φ_{k+1}. Clearly, they are not in class φ_i for $i = 1, \ldots, k$ and therefore $y \in V(G^x_{\overline{\varphi}_i})$ for all $i = 1, \ldots, k$, from what Equation (5.1) and finally surjectivity follows.

It remains to verify the isomorphism property, that is $[V_R(x), V_R(y)]$ is an edge in $G/\mathcal{P}^R$ if and only if $[(G^x_{\overline{\varphi}_1}, \ldots, G^x_{\overline{\varphi}_n}), (G^y_{\overline{\varphi}_1}, \ldots, G^y_{\overline{\varphi}_n})]$ is an edge in $\Box_{i=1}^n G_{\varphi_i}/\mathcal{P}^R_{\overline{\varphi}_i}$. Let $[V_R(x), V_R(y)] \in E(G/\mathcal{P}^R)$, that is, there exists a vertex $x' \in V_R(x)$ and a vertex $y' \in V_R(y)$ such that $[x', y']$ is an edge in G and therefore in φ_i for some i, $1 \leq i \leq n$. This implies $G^x_{\overline{\varphi}_j} = G^{x'}_{\overline{\varphi}_j} = G^{y'}_{\overline{\varphi}_j} = G^y_{\overline{\varphi}_j}$ for all $j \neq i$ and $[G^x_{\overline{\varphi}_i}, G^y_{\overline{\varphi}_i}] \in E(G_{\varphi_i}/\mathcal{P}^R_{\overline{\varphi}_i})$. Thus, $[(G^x_{\overline{\varphi}_1}, \ldots, G^x_{\overline{\varphi}_n}), (G^y_{\overline{\varphi}_1}, \ldots, G^y_{\overline{\varphi}_n})]$ is an edge in $\Box_{i=1}^n G_{\varphi_i}/\mathcal{P}^R_{\overline{\varphi}_i}$. Conversely, let $[(G^x_{\overline{\varphi}_1}, \ldots, G^x_{\overline{\varphi}_n}), (G^y_{\overline{\varphi}_1}, \ldots, G^y_{\overline{\varphi}_n})] \in E(\Box_{i=1}^n G_{\varphi_i}/\mathcal{P}^R_{\overline{\varphi}_i})$. There must be an i, $1 \leq i \leq n$ such that $[G^x_{\overline{\varphi}_i}, G^y_{\overline{\varphi}_i}] \in E(G_{\varphi_i}/\mathcal{P}^R_{\overline{\varphi}_i})$ and $G^x_{\overline{\varphi}_j} = G^y_{\overline{\varphi}_j}$ for all $j \neq i$. $[G^x_{\overline{\varphi}_i}, G^y_{\overline{\varphi}_i}] \in E(G_{\varphi_i}/\mathcal{P}^R_{\overline{\varphi}_i})$ implies that there exists a vertex $x' \in V(G^x_{\overline{\varphi}_i})$ and a vertex $y' \in V(G^y_{\overline{\varphi}_j})$ such that $[x', y'] \in \varphi_i$ in G. From Lemma 3.12, we can conclude that there exists a vertex $z \in V(G^y_{\overline{\varphi}_i})$ such that $[x, z] \in \varphi_i$. This in turn implies $z \in V(G^x_{\overline{\varphi}_j}) = V(G^y_{\overline{\varphi}_j})$ for all $j \neq i$ and thus, $z \in V_R(y)$. Hence, $[V_R(x), V_R(y)]$ is an edge in $G/\mathcal{P}^R$. $\square$



(a) Graph G

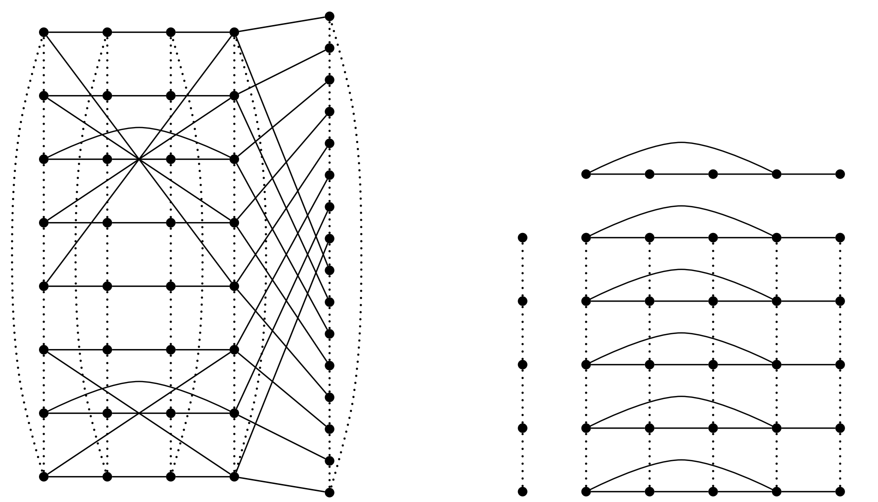
(b) $G_{\varphi_1}/\mathcal{P}^R_{\overline{\varphi}_1}$

(c) $G_{\varphi_2}/\mathcal{P}^R_{\overline{\varphi}_2}$

(d) $G_{\varphi_3}/\mathcal{P}^R_{\overline{\varphi}_3}$

(e) $G/\mathcal{P}^R = \square_{i=1}^{3} G_{\varphi_i}/\mathcal{P}^R_{\overline{\varphi}_i}$

Fig. 5.1: It is shown on the left-hand side a graph G with RSP-relation R on $E(G)$ with equivalence classes φ_1 (solid), φ_2 (dashed), and φ_3 (dotted). We have $\mathcal{P}^R_{\overline{\varphi}_1} = \{\{1,4,7,10\},\{2,5,8,11\},\{3,6,9,12\}\}$, $\mathcal{P}^R_{\overline{\varphi}_2} = \{V(G)\}$, $\mathcal{P}^R_{\overline{\varphi}_3} = \{\{1,2,\ldots,6\},\{7,8,\ldots,12\}\}$ and $\mathcal{P}^R = \{\{1,4\},\{2,5\},\{3,6\},\{7,10\},\{8,11\},\{9,12\}\}$. The corresponding quotient graphs $G_{\varphi_i}/\mathcal{P}^R_{\overline{\varphi}_i}$, $i = 1,2,3$ and the product graph $G/\mathcal{P}^R$ are shown on the right-hand side.



(a)

(b)

Fig. 5.2: **(a)** A graph with an RSP-relation R whose equivalence classes are highlighted by dashed and solid edges. **(b)** The corresponding quotient graph $G/\mathcal{P}^R$ and its Cartesian prime factors.

Corollary 5.2. *Let the conditions of Theorem 5.1 be satisfied. If furthermore $G_{\overline{\varphi}}$ is an induced subgraph of G for all $\varphi \sqsubseteq R$ then*

$$G/\mathcal{P}^R \cong \mathop{\square}_{\varphi \sqsubseteq R} \mathcal{N}(G/\mathcal{P}^R_{\overline{\varphi}}).$$

Proof. It suffices to show that $G/\mathcal{P}^R$ has no loops if all $G_{\overline{\varphi}}$ are induced. We will prove this by contradiction. Therefore, suppose that $G/\mathcal{P}^R$ contains a loop $[V_R(x), V_R(x)]$ for some

$x \in V(G)$. Hence, there are vertices $y, z \in V_R(x)$ with $[y, z] \in E(G)$. Clearly, $[y, z] \in \varphi$ for some $\varphi \sqsubseteq R$. But since $y, z \in V(G^x_{\overline{\varphi}})$ it follows that $G_{\overline{\varphi}}$ is not induced, a contradiction. $\square$

Product structures and equitable partitions are compatible in the following sense:

Proposition 5.3 ([4]). *Let $G = \Box_{i=1}^n G_i$ and let $\mathcal{P}_i$ be an equitable partition on G_i. Then there is an equitable partition $\mathcal{P}$ of G such that*

$$\mathop{\Box}_{i=1}^{n} \overrightarrow{G_i/\mathcal{P}_i} = \overrightarrow{G/\mathcal{P}}.$$

Our next result shows that (equitable) partitions constructed in Section 4.2 arrange themselves as a special case of Proposition 5.3.

Corollary 5.4. *Suppose the conditions of Theorem 5.1 are satisfied. If furthermore R is well-behaved, then*

$$\overrightarrow{G/\mathcal{P}^R} \cong \mathop{\Box}_{\varphi \sqsubseteq R} \overrightarrow{G_\varphi/\mathcal{P}^R_{\overline{\varphi}}}.$$

Proof. Since the underlying undirected and unweighted graphs of $\overrightarrow{G/\mathcal{P}^R}$ and $\overrightarrow{G_\varphi/\mathcal{P}^R_{\overline{\varphi}}}$ are exactly $G/\mathcal{P}^R$ and $G_\varphi/\mathcal{P}^R_{\overline{\varphi}}$, respectively, it suffices to show that the weights are transferred as in Equation (2.1). This follows immediately from Lemma 4.18 and Lemma 4.14 and the fact that $|N_G(x) \cap V_R(y)| = \sum_{\varphi \sqsubseteq R} |N_\varphi(x) \cap V_R(y)|$. $\square$

With the help of the results obtained in this section we can strengthen a useful result of [47]:

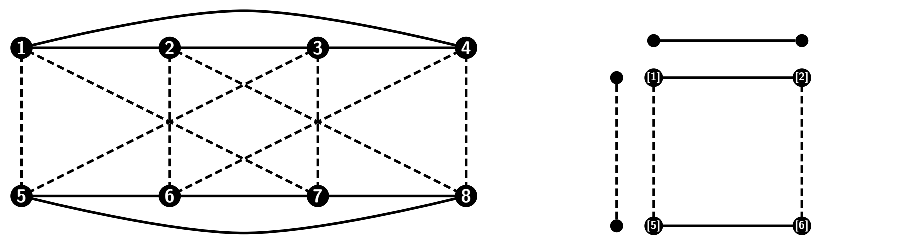
Proposition 5.5. *Let Q be an RSP-relation on the edge set $E(G)$ of a connected graph G and let $\varphi \sqsubseteq Q$. Then $|V(G^x_\varphi) \cap V(G^y_{\overline{\varphi}})| = 1$ holds for all $x, y \in V(G)$ if and only if $R = \{\varphi, \overline{\varphi}\}$ is a product relation.*

Proof. It has been shown in [47] that for product relations, i.e., convex equivalence relations satisfying the square property, holds $|V(G^x_\varphi) \cap V(G^y_{\overline{\varphi}})| = 1$ for all $x, y \in V(G)$. Since any convex RSP-relation already has the square property (see Section 3.2), it remains to show, therefore, that the converse is also true.

Notice that $\overline{\overline{\varphi}} = \varphi$. Hence, the partition induced by R is $\mathcal{P}^R = \{V(G^x_\varphi) \cap V(G^x_{\overline{\varphi}}) \mid x \in V(G)\}$. By assumption, $\mathcal{P}^R$ consists exclusively of singletons. Thus $G = G/\mathcal{P}^R$. Recall that $\mathcal{P}^R_{\overline{\varphi}} = \{V(G^x_{\overline{\varphi}}) \mid x \in V(G)\}$ and $\mathcal{P}^R_\varphi = \{V(G^x_\varphi) \mid x \in V(G)\}$ are the partitions of the graphs G_φ and $G_{\overline{\varphi}}$ respectively. For arbitrary $y \in V(G)$ let $\mathcal{P}^R_{\overline{\varphi}}(y)$ denote the restriction of $\mathcal{P}_{\overline{\varphi}}$ to the connected component G^y_φ of G_φ, that is $\mathcal{P}^R_{\overline{\varphi}}(y) = \{V(G^x_{\overline{\varphi}}) \cap V(G^y_\varphi) \mid x \in V(G)\}$. From Lemma 3.13, Lemma 3.12 and the definition of the quotient graphs, we can conclude that the mapping $G^x_{\overline{\varphi}} \cap G^y_\varphi \mapsto G^x_{\overline{\varphi}}$ defines an isomorphism $G_\varphi/\mathcal{P}^R_{\overline{\varphi}} \cong G^y_\varphi/\mathcal{P}^R_{\overline{\varphi}}(y)$ for all $y \in V(G)$ and since $|V(G^x_\varphi) \cap V(G^y_{\overline{\varphi}})| = 1$ holds for all $x, y \in V(G)$, we even have $G_\varphi/\mathcal{P}^R_{\overline{\varphi}} \cong G^y_\varphi$. Analogously, it follows $G_{\overline{\varphi}}/\mathcal{P}^R_{\overline{\varphi}} \cong G^y_{\overline{\varphi}}$ for all $y \in V(G)$. Thus, $G \cong G^x_\varphi \,\Box\, G^y_{\overline{\varphi}}$ for all $x, y \in V(G)$, demonstrating that $R = \{\varphi, \overline{\varphi}\}$ is a product relation. $\square$

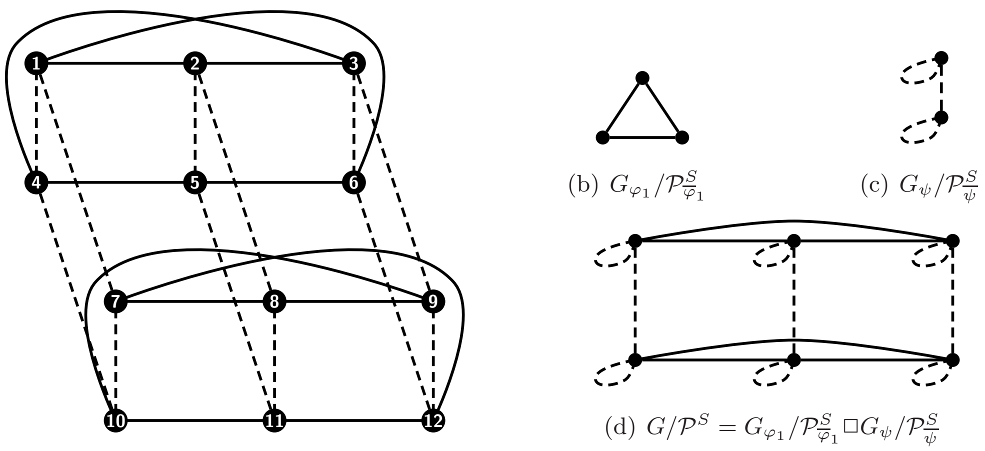
Refinements and Coarse Graining

Given a graph G and a nontrivial RSP-relation R on $E(G)$ it will often be the case that $G/\mathcal{P}^R$ has no "real" product structure, as in the example of Figure 3.1. Here, $V(G_{\overline{\varphi}_i}) = V(G)$ for each of the four equivalence classes, so that $G/\mathcal{P}^R$ is the trivial graph $\mathcal{L}K_1$ consisting of a single vertex with a loop. In Section 2.3, we have shown that a coarse graining S of an RSP-relation R in general leads to a refinement $\mathcal{P}^S$ of the vertex partition $\mathcal{P}^R$. Hence, we can expect to obtain larger quotient graph $G/\mathcal{P}^S$ with a "richer" product structure. This is indeed sometimes the case as shown by the example in Fig. 5.3. However, a coarser relation $S \supseteq R$ does not always lead to a partition $\mathcal{P}^S$ that is strictly finer than $\mathcal{P}^R$, see Fig. 5.4 for an example. In the following, we therefore explore the conditions under which a *strictly* finer partition $\mathcal{P}^S$ of the vertex set is obtained by a coarser equivalence relation $S \supseteq R$.



(a) Graph G (b) $G/\mathcal{P}^R = G_{\psi_1}/\mathcal{P}^R_{\psi_1} \Box G_{\psi_2}/\mathcal{P}^R_{\psi_2}$

Fig. 5.3: The coarse graining $R = \{\psi_1 = \varphi_1 \cup \varphi_2, \psi_2 = \varphi_3 \cup \varphi_4\}$ obtained from the equivalence relation Q of Fig. 3.1 generates the quotient graph $G/\mathcal{P}^R \cong K_2 \Box K_2$ with non-trivial product structure.



(a) Graph G (b) $G_{\varphi_1}/\mathcal{P}^S_{\varphi_1}$ (c) $G_{\psi}/\mathcal{P}^S_{\psi}$ (d) $G/\mathcal{P}^S = G_{\varphi_1}/\mathcal{P}^S_{\varphi_1} \Box G_{\psi}/\mathcal{P}^S_{\psi}$

Fig. 5.4: The coarse graining $S = \{\varphi_1, \psi = \varphi_2 \cup \varphi_3\}$ of the relation R of Fig. 5.1 leads to the same partition $\mathcal{P}^S = \mathcal{P}^R$ of $V(G)$ and thus to identical quotient graphs.

Lemma 5.6. *Let φ and ψ be two equivalence classes of an RSP-relation R on the edge set $E(G)$ of a connected graph G. Then for all $x \in V(G)$ holds*

$$V(G^x_{\varphi\cup\psi}) = \bigcup_{y\in V(G^x_\varphi)} V(G^y_\psi) = \bigcup_{y\in V(G^x_\psi)} V(G^y_\varphi).$$

Proof. It suffices to show the first equation. Therefore, let $z \in \bigcup_{y\in V(G^x_\varphi)} V(G^y_\psi)$, that is, there exists a vertex $y' \in V(G^x_\varphi)$ such that $z \in V(G^{y'}_\psi)$. Hence, there is a path $P_{x,y'}$ from x to y' in φ and a path $P_{y',z}$ from y' to z in ψ. Thus, $P_{x,y'} \cup P_{y',z}$ is a path from x to z in $\varphi\cup\psi$ and therefore $z \in V(G^x_{\varphi\cup\psi})$ from which we can conclude $\bigcup_{y\in V(G^x_\varphi)} V(G^y_\psi) \subseteq V(G^x_{\varphi\cup\psi})$.

Now, let $z \in V(G^x_{\varphi\cup\psi})$. Clearly, the restriction of R to $G^x_{\varphi\cup\psi}$ is an RSP-relation on $E(G^x_{\varphi\cup\psi})$ with only two equivalence classes φ and ψ. Therefore, by Lemma 3.13 we can conclude that $V(G^z_\psi) \cap V(G^x_\varphi) \neq \emptyset$. Let $y \in V(G^z_\psi) \cap V(G^x_\varphi)$. It follows that $G^z_\psi = G^y_\psi$ and thus, $z \in \bigcup_{y\in V(G^x_\varphi)} V(G^y_\psi)$ since in particular $y \in V(G^x_\varphi)$. From $V(G^x_{\varphi\cup\psi}) \subseteq \bigcup_{y\in V(G^x_\varphi)} V(G^y_\psi)$ we conclude equality of the sets. □

Lemma 5.7. *Let R be an RSP-relation on the edge set $E(G)$ of a connected graph G and let $\varphi, \psi \sqsubseteq R$, $\varphi \neq \psi$. Then $V(G^x_{\overline{\varphi}})\cap V(G^x_{\overline{\psi}}) = V(G^x_{\overline{\varphi\cup\psi}})$ if and only if $V(G^x_\varphi)\cap V(G^x_{\overline{\varphi}}) \subseteq V(G^x_{\overline{\varphi\cup\psi}})$.*

Proof. From Lemma 5.6 we can compute $V(G^x_{\overline{\varphi}})\cap V(G^x_{\overline{\psi}}) = V(G^x_{\overline{\varphi}})\cap V(G^x_{\varphi\cup(\overline{\varphi\cup\psi})}) = V(G^x_{\overline{\varphi}})\cap \Big(\bigcup_{w\in V(G^x_\varphi)} V(G^w_{\overline{\varphi\cup\psi}})\Big) = \bigcup_{w\in V(G^x_\varphi)} \Big(V(G^x_{\overline{\varphi}}) \cap V(G^w_{\overline{\varphi\cup\psi}})\Big)$.

Notice that $V(G^v_{\overline{\varphi\cup\psi}}) \subseteq V(G^x_{\overline{\varphi}})$ if and only if $v \in V(G^x_{\overline{\varphi}})$, otherwise we would have $V(G^x_{\overline{\varphi}}) \cap V(G^v_{\overline{\varphi\cup\psi}}) = \emptyset$. Therefore,

$V(G^x_{\overline{\varphi}}) \cap V(G^x_{\overline{\psi}}) = \bigcup_{w\in V(G^x_\varphi)\cap V(G^x_{\overline{\varphi}})} V(G^w_{\overline{\varphi\cup\psi}}) = V(G^x_{\overline{\varphi\cup\psi}}) \mathbin{\dot\cup} \Big(\bigcup_{w\in\mathcal{X}} V(G^w_{\overline{\varphi\cup\psi}})\Big)$ with $\mathcal{X} = V(G^x_\varphi) \cap V(G^x_{\overline{\varphi}}) \setminus V(G^x_{\overline{\varphi\cup\psi}})$

Hence, we have $V(G^x_{\overline{\varphi}}) \cap V(G^x_{\overline{\psi}}) = V(G^x_{\overline{\varphi\cup\psi}})$ if and only if $\mathcal{X} = \emptyset$ which is equivalent to $V(G^x_\varphi) \cap V(G^x_{\overline{\varphi}}) \subseteq V(G^x_{\overline{\varphi\cup\psi}})$. □

Lemma 5.8. *Let R be an RSP-relation on the edge set $E(G)$ of a connected graph G with two distinct equivalence classes $\varphi, \psi \sqsubseteq R$.*

(1) If $V(G^x_\varphi) \subseteq V(G^x_\psi)$ for some $x \in V(G)$ then $V(G^y_\varphi) \subseteq V(G^x_\psi)$ holds for all $y \in V(G^x_\psi)$.

(2) If $V(G^x_\varphi) \subseteq V(G^x_{\overline{\varphi}})$ for some $x \in V(G)$ then $V(G^x_{\overline{\varphi}}) = V(G)$.

(3) If $V(G^x_{\overline{\varphi}}) = V(G)$, $x \in V(G)$, then for all $y \in V(G)$ holds $V(G^y_\varphi) \cap V(G^y_{\overline{\varphi}}) \subseteq V(G^y_{\overline{\varphi\cup\psi}})$ if and only if $V(G^y_\varphi) \subseteq V(G^y_{\overline{\varphi\cup\psi}})$.

Proof. (1) Let $X := \Big\{v \in V(G^x_\psi) \mid V(G^v_\varphi) \subseteq V(G^x_\psi)\Big\}$. If $V(G^x_\varphi) \subseteq V(G^x_\psi)$ then $X \neq \emptyset$. Suppose $V(G^x_\psi) \setminus X \neq \emptyset$. By connectedness of G^x_ψ, there exist some vertices $y \in V(G^x_\psi) \setminus X$ and $v \in X$ such that $[v, y] \in \psi$. Clearly, $y \notin V(G^v_\varphi)$. Since $y \notin X$, there exists a vertex $w \in V(G^y_\varphi) \setminus V(G^x_\psi)$. Since $[v, y] \in \psi$, we can use Lemma 3.12 to conclude that there exists a vertex $z \in V(G^v_\varphi)$ such that $[z, w] \in \psi$. This implies $w \in V(G^z_\psi) = V(G^x_\psi)$, since $v \in X$, a contradiction.

(2) Let $V(G^x_\varphi) \subseteq V(G^x_{\overline{\varphi}})$ and suppose $V(G) \setminus V(G^x_{\overline{\varphi}}) \neq \emptyset$. By connectedness of G, there exist vertices $v \in V(G^x_{\overline{\varphi}})$ and $y \in V(G) \setminus V(G^x_{\overline{\varphi}})$ such that $[v, y] \in E(G)$. Obviously, $[v, y]$ must be in φ. Hence, $y \in V(G^w_\varphi)$. From the first assertion, we conclude that this implies $y \in V(G^x_{\overline{\varphi}})$, a contradiction.

(3) Clear. $\square$

We conclude our presentation by summarizing conditions under which the joining of two equivalence classes of a RSP-relation does not affect the partitioning of the vertex set.

Corollary 5.9. *Let R be an RSP-relation on the edge set $E(G)$ of a connected graph G with two distinct equivalence classes $\varphi, \psi \sqsubseteq R$ and denote by S the RSP-relation obtained from R by joining φ and ψ. Then:*

(1) $\mathcal{P}^R = \mathcal{P}^S$ if φ or ψ belong to a factor of G.

(2) If there is a vertex $x \in V(G)$ with $V(G^x_\varphi) \subseteq V(G^x_{\overline{\varphi}})$ then $\mathcal{P}^R = \mathcal{P}^S$ if and only if $V(G^y_\varphi) \subseteq V(G^y_{\overline{\varphi\cup\psi}})$ holds for all $y \in V(G)$.

(3) If $\mathcal{P}^R = \mathcal{P}^S$ then $\varphi \cup \psi$ belongs to a factor of G if and only if both φ and ψ belong to a factor of G.

Proof. (1) W.l.o.g., let φ correspond to a factor of G. Then it holds $V(G^x_\varphi) \cap V(G^x_{\overline{\varphi}}) = \{x\} \subseteq V(G_{\overline{\varphi\cup\psi}})$ for all $x \in V(G)$, which implies the assertion.

(2) Follows immediately from Lemma 5.8.

(3) If both φ and ψ correspond to factors, then clearly $\varphi \cup \psi$ also corresponds to a factor. Conversely, suppose $\varphi \cup \psi$ correspond to a factor and suppose $\mathcal{P}^R = \mathcal{P}^S$. Then $|V(G^x_{\varphi\cup\psi}) \cap V(G^x_{\overline{\varphi\cup\psi}})| = 1$ and $V(G^x_\varphi) \cap V(G^x_{\overline{\varphi}}) \subseteq V(G_{\overline{\varphi\cup\psi}})$ holds for all $x \in V(G)$. Note that $V(G^x_\varphi) \subseteq V(G^x_{\varphi\cup\psi})$ and hence $V(G^x_\varphi) \cap V(G^x_{\overline{\varphi}}) = (V(G^x_\varphi) \cap V(G^x_{\overline{\varphi}})) \cap V(G^x_{\varphi\cup\psi}) \subseteq V(G_{\overline{\varphi\cup\psi}}) \cap V(G^x_{\varphi\cup\psi})$ holds for all $x \in V(G)$. This implies $V(G^x_\varphi) \cap V(G^x_{\overline{\varphi}}) = \{x\}$ for all $x \in V(G)$. By Proposition 5.5 we can now conclude that φ belongs to a factor of G. Analogously, it follows that ψ belongs to a factor of G. $\square$

5.2 RSP-relations on Product Graphs

Graph products are intimately related with the (relaxed) square property. It seems natural therefore, to ask whether finest RSP-relations can be found more easily in products. We use the symbol $\circledast$ for one of the three graph products defined in Section 2.2.

Definition 5.10 (Product of Relations)**.** *For $\circledast \in \{\square, \boxtimes, \times\}$ let $G = \circledast_{i\in I} G_i$. For each $i \in I$ let R_i be an equivalence relation on $E(G_i)$. Furthermore, define for $e \in E(G)$ the set $I_e := \{i \in I \mid p_i(e) \in E(G_i)\}$. We define an equivalence relation $\circledast_{i\in I} R_i$ on $E(G)$ as follows: $(e, f) \in \circledast_{i\in I} R_i$ if and only if $I_e = I_f$ and $(p_i(e), p_i(f)) \in R_i$, for all $i \in I_e$.*

If $\circledast = \square$ then $|I_e| = 1$ for all $e \in E(G)$, and if $\circledast = \times$ then $I_e = I$ for all $e \in E(G)$.

Lemma 5.11. *For $\circledast \in \{\Box, \boxtimes, \times\}$ let $G = \circledast_{i\in I} G_i$. For each $i \in I$ let R_i be an equivalence relation on $E(G_i)$. Then $R := \circledast_{i\in I} R_i$ is an RSP-relation if and only if R_i is an RSP-relation for all $i \in I$.*

Proof. First assume R_i has the relaxed square property for all $i \in I$. We have to show that R has the relaxed square property. Therefore, let $e = [x, y], f = [x, z] \in E(G)$ such that $(e, f) \notin R$. We need to show that there exists a vertex $w \in V(G)$ such that $e' = [w, z] \in E(G)$, $f' = [w, y] \in E(G)$ and $(e, e') \in R$ as well as $(f, f') \in R$.

Let $I_0 := \{i \in I \mid (p_i(e), p_i(f)) \in R_i\}$. Notice, that $I_0 \subseteq I_e \cap I_f$. Moreover, we have $(p_j(e), p_j(f)) \notin R_j$ for all $j \in (I_e \cap I_f) \setminus I_0 =: I^*$. Since R_i has the relaxed square property for all $i \in I$, for all $j \in I^*$ there exists a vertex $w_j \in V(G_j)$ such that $(p_j(e), [p_j(z), w_j]) \in R_j$ as well as $(p_j(f), [p_j(y), w_j]) \in R_j$.

Let $w \in V(G)$ such that

$$\begin{aligned} p_i(w) &= p_i(x), && \text{for all } i \in I_0, \text{ and} \\ p_i(w) &= w_i, && \text{for all } i \in I^*, \text{ and} \\ p_i(w) &= p_i(z), && \text{for all } i \in I \setminus I_e, \text{ and} \\ p_i(w) &= p_i(y), && \text{for all } i \in I \setminus I_f. \end{aligned}$$

Since $I = I_0 \dot\cup I^* \dot\cup (I \setminus (I_e \cap I_f))$, $I \setminus (I_e \cap I_f) = I \setminus I_e \cup I \setminus I_f$ and $p_i(z) = p_i(x) = p_i(y)$ for all $i \in I \setminus I_e \cap I \setminus I_f$, this vertex exists in $V(G)$ and is well defined.

We now have to verify that w has the desired properties. More precisely, we have to verify the following statements:

(i) $p_i(w) = p_i(z)$ for all $i \in I \setminus I_e$,

(ii) $p_i(w) = p_i(y)$ for all $i \in I \setminus I_f$,

(iii) $e'_i := [p_i(z), p_i(w)] \in E(G_i)$ and $(p_i(e), e'_i) \in R_i$ for all $i \in I_e$,

(iv) $f'_i := [p_i(y), p_i(w)] \in E(G_i)$ and $(p_i(f), f'_i) \in R_i$ for all $i \in I_f$.

Assertions (i) and (ii) are trivially fulfilled by construction. To prove assertion (iii), note it holds that $I_e = I_0 \dot\cup I^* \dot\cup (I_e \setminus I_f)$. From $p_i(w) = p_i(x)$ for all $i \in I_0$, we conclude $e'_i = [p_i(z), p_i(x)] = p_i(f) \in E(G_i)$, and moreover, by construction of I_0 and since R_i is an equivalence relation, we have $(p_i(e), e'_i) \in R_i$ for all $i \in I_0$. By the choice of w, it holds that $e'_i \in E(G_i)$ and $(p_i(e), e'_i) \in R_i$ for all $i \in I^*$. Finally, we have $e'_i = [p_i(z), p_i(y)] = [p_i(x), p_i(y)] = p_i(e) \in E(G_i)$ for all $i \in I_e \setminus I_f$ and since R_i is an equivalence relation, $(e'_i, p_i(e)) \in R_i$. Thus, $e' = [w, z] \in E(G)$ and $(e, e') \in R$.

Assertion (iv), which implies $f' = [w, y] \in E(G)$ and $(f, f') \in R$, can be shown analogously.

Now assume R is an RSP-relation. We have to show that for all $i \in I$, R_i has the relaxed square property. Therefore, let $i \in I$ and $e_i = [x_i, y_i], f_i = [x_i, z_i]$ be two adjacent edges in G_i

such that $(e_i, f_i) \notin R_i$. We need to show, that there exists some vertex $w_i \in V(G_i)$ such that $e'_i := [w_i, z_i], f'_i := [w_i, y_i]$ are edges in G_i with $(e_i, e'_i) \in R_i$ and $(f_i, f'_i) \in R_i$. By definition of $\circledast$, there exist edges $e = [x, y], f = [x, z] \in E(G)$, $p_i(x) = x_i, p_i(y) = y_i, p_i(z) = z_i$, with $p_i(e) = e_i$ and $p_i(f) = f_i$, that are adjacent. It holds that $i \in I_e \cap I_f$ and by definition of R, $(e, f) \notin R$. Since R has the relaxed square property, there exists some vertex $w \in V(G)$ such that $e' := [w, z], f' := [w, y]$ are edges in G with $(e, e') \in R$ and $(f, f') \in R$. That is, by definition of R, $I_e = I_{e'}$ and $(p_j(e), p_j(e')) \in R_j$ for all $j \in I_e$ as well as $I_f = I_{f'}$ and $(p_j(f), p_j(f')) \in R_j$ for all $j \in I_f$. Thus, we have in particular $(e_i, p_i(e')), (f_i, p_i(f')) \in R_i$ and $z_i \neq p_i(w) \neq y_i$. Moreover, $p_i(w) \neq x_i$, since otherwise $p_i(e') = [p_i(w), p_i(z)] = [x_i, z_i] = f_i$ and therefore $(f_i, e_i) = (p_i(e'), p_i(e)) \in R_i$ must hold, a contradiction. Hence, with $w_i := p_i(w)$ the assertion follows. □

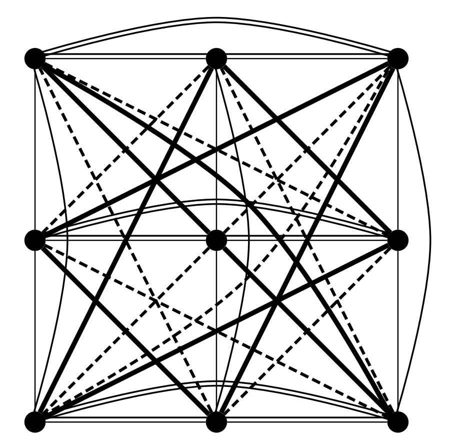


Fig. 5.5: Refinement of product of relations of K_9 w.r.t. $K_9 \cong K_3 \boxtimes K_3$

For $\circledast \in \{\times, \boxtimes\}$, the relation $R = \circledast_{i \in I} R_i$ need not be the finest RSP-relation on $E(G) = E(\circledast_{i \in I} G_i)$ although R_i is a finest RSP-relation on $E(G_i)$ for all $i \in I$. See Fig. 5.5 for an example: Shown is the complete graph K_9 with a finest RSP-relation consisting of four equivalence classes depicted by drawn-through, double, dashed and thick lines. Joining the two classes with dashed and thick edges to one class, one gets a coarser relation $R_1 \boxtimes R_2$, w.r.t. $K_9 \cong K_3 \boxtimes K_3$ where R_i denotes the trivial relation on $E(K_3)$. This together with Lemma 3.6 implies that also $R_1 \times R_2$ is not a finest RSP-relation on $E(K_3 \times K_3)$.

However, this does not hold for the Cartesian product $\Box$. Moreover, we have:

Lemma 5.12. *Let $G = \Box_{i \in I} G_i$ be a connected and simple graph. Then R is a finest RSP-relation on $E(G)$ if and only if $R = \Box_{i \in I} R_i$ where each R_i is a finest RSP-relation on $E(G_i)$.*

Proof. First, observe the following: Let R' be an arbitrary RSP-relation on G and $[x, y], [y, z] \in E(G)$ incident edges that lie in the same layer of G, i.e. $p_j([x, y]) \in E(G_j)$ and $p_j([y, z]) \in E(G_j)$ for some $j \in I$. Moreover, let $[x, y]$ and $[y, z]$ be in different equivalence classes of R'. Since R' is an RSP-relation, they lie on a four cycle $x - y - z - w$ with opposite edges in the same equivalence class. By the definition of the Cartesian product, w is also in the same layer as x, y, z, that is, $w \in V(G_j^y)$. This shows that R' limited to the subgraph G_j^y is also an RSP-relation.

Let now $[x, y], [w, z] \in E(G)$ be such edges that lie on a four cycle $x - y - z - w$ with $j \in I$ such that $p_j([x, y]) = p_j([w, z]) \in E(G_j)$. Assume that $[x, y]$ and $[w, z]$ do not lie in the same equivalence class of R'. Then at least one of the pairs $[x, y], [x, w]$ or $[w, z], [x, w]$ does not lie in the same equivalence class of R'. Without loss of generality let $[x, y]$ and $[x, w]$ lie in different equivalence classes of R'. By the definition of the Cartesian product, $x - y - z - w$ is the only four cycle that contains $[x, y]$ and $[x, w]$. Since R' is an RSP-relation, $[x, y]$ and $[w, z]$ lie in the same equivalence class. By connectedness of G, all layers are connected. Therefore, all edges $\{[a, b] \in E(G) : p_j([a, b]) = p_j([x, y])\}$ are in the same equivalence class.

Assume now that R is a finest RSP-relation on G. We define a relation R_j on $E(G_j)$ for every $j \in I$ by $(e, f) \in R_j$ for $e, f \in E(G_j)$ if $p_j(e') = e$, $p_j(f') = f$ for some $e', f' \in E(G)$ and $(e', f') \in R$. By above arguments, this is an RSP-relation on G_j. Notice that R corresponds to $\Box_{i \in I} R_i$ with possibly some joint equivalence classes, that emerge from different layers of $\Box_{i \in I} G_i$. Since R is a finest RSP-relation, $R = \Box_{i \in I} R_i$. If R_j is not a finest RSP-relation on G_j for some $j \in I$, then the product of a finer relation on G_j with $\Box_{i \in I \setminus \{j\}} R_i$ is a finer relation as R, a contradiction.

To see the converse, let $R = \Box_{i \in I} R_i$, where R_i is a finest RSP-relation on G_i. If Q is a finest relation on G, that is finer than R, by above arguments, $Q = \Box_{i \in I} Q_i$, where Q_i is finer or equal than R_i for every $i \in I$. Thus $Q = R$. $\square$

Lemma 5.12 implies not only that $R = \Box_{i \in I} R_i$ is a finest RSP-relation on $E(G) = E(\Box_{i \in I} G_i)$ if R_i is a finest RSP-relation on $E(G_i)$, but also that any (finest) RSP-relation on a Cartesian product graph must reflect the layers w.r.t. its (prime) factorization. However, this is not true for $\circledast = \boxtimes$, as an example take $K_6 \cong K_3 \boxtimes K_2$ with the relation defined in Lemma 3.20.

The next lemma shows, how the structure of the quotient graphs from graph products can be derived from the structure of the quotients of its factors.

Lemma 5.13. *For $i \in I$ let G_i be connected graphs and let R_i be an RSP-relation on the edge set $E(G_i)$. The following hold.*

($\Box$) *If $G = \Box_{i \in I} G_i$ and $R = \Box_{i \in I} R_i$ then $G/\mathcal{P}^R = \Box_{i \in I} G_i/\mathcal{P}^{R_i}$.*

($\boxtimes$) *If $G = \boxtimes_{i \in I} G_i$ and $R = \boxtimes_{i \in I} R_i$ then $G/\mathcal{P}^R = \mathcal{L}K_1$.*

Proof. ($\Box$) By construction, ψ is an equivalence class of R if and only if there exists an $i \in I$ such that $p_i(e) \in E(G_i)$ and there exists $\varphi \in R_i$ with $p_i(e) \in \varphi$ for all $e \in \psi$. Hence, there exists a bijection $R = \circledast_{i \in I} R_i \to \dot{\bigcup}_{i \in I} R_i$. For $i \in I$ let $\varphi_1^i, \dots, \varphi_{n_i}^i$ be the equivalence classes of R_i. Moreover, for $i \in I$ and $1 \leq j \leq n_i$ let ψ_j^i be the equivalence class of R such that $I_e = \{i\}$ and $p_i(e) \in \varphi_j^i$ for all $e \in \psi_j^i$. Thus, with Theorem 5.1, we obtain

$G/\mathcal{P}^R = \Box_{\psi \sqsubseteq R} G_\psi/\mathcal{P}^R_{\overline{\psi}} = \Box_{i\in I}(\Box_{j=1}^{n_i} G_{\psi^i_j}/\mathcal{P}^R_{\overline{\psi^i_j}})$. Furthermore, due to Theorem 5.1, we have $\Box_{i\in I} G_i/\mathcal{P}^{R_i} = \Box_{i\in I}(\Box_{j=1}^{n_i} G_{i\varphi^i_j}/\mathcal{P}^{R_i}_{\overline{\varphi^i_j}})$.

Hence, we need to show $G_{\psi^i_j}/\mathcal{P}^R_{\overline{\psi^i_j}} \cong G_{i\varphi^i_j}/\mathcal{P}^{R_i}_{\overline{\varphi^i_j}}$ for all $i \in I$ and $1 \leq j \leq n_i$, to prove the assertion. Therefore, we show that $G_{\overline{\psi^i_j}}(x) \mapsto G_{i\overline{\varphi^i_j}}(p_i(x))$ for all $x \in V(G)$ defines an isomorphism $G_{\psi^i_j}/\mathcal{P}^R_{\overline{\psi^i_j}} \cong G_{i\varphi^i_j}/\mathcal{P}^{R_i}_{\overline{\varphi^i_j}}$. If $G_{\overline{\psi^i_j}}(x) = G_{\overline{\psi^i_j}}(y)$, there exists a path $P_{x,y} := (e_1, \ldots, e_k)$ from x to y in G, such that $e_l \notin \psi^i_j$ for $1 \leq l \leq k$. Then $p_i(P_{x,y}) = (p_1(e_1), \ldots, p_i(e_k))$ is a walk from $p_i(x)$ to $p_i(y)$ in G_i and by construction, it holds that $p_i(e_l) \notin \varphi^i_j$ for $1 \leq l \leq k$, i.e., $G_{i\overline{\varphi^i_j}}(p_i(x)) = G_{i\overline{\varphi^i_j}}(p_i(y))$. Thus, this mapping is well defined. Moreover, by the projection properties of a Cartesian product into its factors, this mapping is surjective. Now, suppose $G_{i\overline{\varphi^i_j}}(p_i(x)) = G_{i\overline{\varphi^i_j}}(p_i(y))$, i.e., there exists a path $P_{p_i(x),p_i(y)} := (e_1, \ldots, e_k)$ from $p_i(x)$ to $p_i(y)$ in G_i such that $e_l \notin \varphi^i_j$ for $1 \leq l \leq k$. Let $w \in V(G)$ s.t. $p_i(w) = p_i(y)$ and $p_r(w) = p_r(x)$ for all $r \in I$, $r \neq i$. Hence, $w \in V(G^x_i)$. Thus, there exists a path $P'_{x,w} = (e'_1, \ldots, e'_k)$ in G with $p_i(e'_l) = e_l$ which implies $e'_l \notin \psi^i_j$ for $1 \leq l \leq k$ and thus $G_{\overline{\psi^i_j}}(x) = G_{\overline{\psi^i_j}}(w)$. Furthermore, by the properties of the Cartesian product, there exists a path $P''_{w,y} = (e''_1, \ldots, e''_s)$ from w to y in G such that $|p_i(e''_l)| = 1$ for $1 \leq l \leq s$, which implies $I_{e''_l} \neq \{i\}$ and consequently $e''_l \notin \psi^i_j$ for $1 \leq l \leq k$. Thus, $G_{\overline{\psi^i_j}}(y) = G_{\overline{\psi^i_j}}(w) = G_{\overline{\psi^i_j}}(x)$, that is, this mapping is injective and therefore bijective. It remains to show that $[G_{\overline{\psi^i_j}}(x), G_{\overline{\psi^i_j}}(y)]$ is an edge in $G_{\psi^i_j}/\mathcal{P}^R_{\overline{\psi^i_j}}$ if and only if $[G_{i\overline{\varphi^i_j}}(p_i(x)), G_{i\overline{\varphi^i_j}}(p_i(y))]$ is an edge in $G_{i\varphi^i_j}/\mathcal{P}^{R_i}_{\overline{\varphi^i_j}}$. By definition, $[G_{\overline{\psi^i_j}}(x), G_{\overline{\psi^i_j}}(y)]$ is an edge in $G_{\psi^i_j}/\mathcal{P}^R_{\overline{\psi^i_j}}$ if and only if there exists $x' \in V(G_{\overline{\psi^i_j}}(x)), y' \in V(G_{\overline{\psi^i_j}}(x))$ s.t. $[x', y'] \in \psi^i_j$, which, by the preceding and by construction, is equivalent to $p_i(x') \in V(G_{i\overline{\varphi^i_j}}(p_i(x))), p_i(y') \in V(G_{i\overline{\varphi^i_j}}(p_i(y)))$ and $[p_i(x'), p_i(y')] \in \varphi^i_j$, from what the assertion follows.

($\boxtimes$) To prove the assertion, we have to show that the spanning subgraph $G_{\overline{\varphi}}$ is connected for all $\varphi \sqsubseteq R$. For each $\varphi \sqsubseteq R$ it holds that $I_e = I_f$ for all $e, f \in \varphi$. We set $I_\varphi := I_e$ for some $e \in \varphi$. Moreover, define $\Phi := \{\psi \sqsubseteq R \mid I_\psi = I_\varphi\}$ Then for $\alpha := \bigcup_{\psi\in\Phi} \psi$, $G_{\overline{\alpha}}$ is a spanning subgraph of $G_{\overline{\varphi}}$. Therefore, it suffices to show that $G_{\overline{\alpha}}$ is connected. To be more precise, we have to show that for all $x, y \in V(G)$, there exists a walk $W_{x,y}$ from x to y in G such that for all $e \in E(W_{x,y})$ it holds that $I_e \neq I_\varphi$.

First, assume $|I_\varphi| > 1$. Since $\Box_{i\in I} G_i$ is a connected spanning subgraph of $\boxtimes_{i\in I} G_i$, there exists a walk $W_{x,y}$ from x to y in $\Box_{i\in I} G_i$. Then for all $e \in E(W_{x,y})$ it holds that $|I_e| = 1$ and thus, $I_e \neq I_\varphi$.

Now, let $|I_\varphi| = 1$, i.e., $I_\varphi = \{j\}$ for some $j \in I$. If $p_j(x) = p_j(y)$, then $y \in V((\Box_{i\in I\setminus\{j\}} G_i)^x)$. In this case, there exists a walk $W_{x,y}$ from x to y in $(\Box_{i\in I\setminus\{j\}} G_i)^x$ that has the desired properties. If $p_j(x) \neq p_j(y)$, let $y' \in V(G)$ such that $p_i(y') = p_i(x)$ for all $i \neq j$ and $p_j(y) = p_j(y')$. Then, as in the previous case, there exists a walk $W_{y,y'}$ from y to y' in $(\Box_{i\in I\setminus\{j\}} G_i)^y$ and hence $I_e \neq \{j\}$ for all $e \in E(W_{y,y'})$. By choice of y', it holds that $y' \in V(G^x_j)$. Let $P_{x,y'} := (x = x_0, x_1, \ldots, x_k = y')$ be a walk from x to y' that is entirely contained in G^x_i. Moreover, for arbitrary $i \in I$ with $i \neq j$ let $z \in V(G^x_i)$ such that $[p_i(x), p_i(z)] \in E(G_i)$ and let $w \in V(G^z_j)$ such that $p_j(w) = p_j(z)$. Then there exists a walk

$P_{z,w} := (z = z_0, z_1, \dots, z_k = w)$ from z to w in G_j^z such that $p_j(x_r) = p_j(z_r)$ for all $0 \leq r \leq k$. By definition of $\boxtimes$, $W_{x,y'} := (x_0, z_1, x_1, z_2, x_2, z_3, \dots, x_{k-1}, z_k = w, x_k = y')$ is a walk from x to y' in G and for the edges $e \in E(W_{x,y'})$ it holds that $I_e = \{i,j\} \neq \{j\} = I_\varphi$ if e is of the form $[x_i, z_{i+1}]$, $0 \leq i \leq k-1$ and $I_e = \{i\} \neq \{j\} = I_\varphi$ if e is of the form $[x_i, z_i]$, $0 \leq i \leq k$. Hence, $W_{x,y} = W_{x,y'} \cup W_{y',y}$ is a walk from x to y that has the desired properties. □

In contrast to the Cartesian and strong products, no general statement can be obtained for the direct product $G = \times_{i\in I} G_i$ of graphs G_i since the structure of direct products strongly depends on additional properties such as bipartiteness.

Lemma 5.13 implies that in case of the Cartesian product, graph multiplication commutes with computation of quotient graphs w.r.t. RSP-relations. In the case of the strong product, the respective quotient graph is independent of the structure of the factors.

Part II

PRODUCT STRUCTURES IN HYPERGRAPHS

Chapter 6

Basics II

Hypergraphs are a natural generalization of undirected graphs in which "edges" may consist of more than 2 vertices. Within this chapter, we provide basic definitions for hypergraphs, mainly following the notation in [5]. We proceed with a short overview about some hypergraph products, namely the Cartesian product of hypergraphs, that generalizes the Cartesian graph product, as well as the normal product and the strong product of hypergraphs, both generalizing the strong graph product.

6.1 Hypergraphs

A *(finite) hypergraph* $H = (V, E)$ consists of a (finite) set V and a collection E of non-empty subsets of V.

The elements of V are called *vertices* and the elements of E are called *hyperedges*, or simply *edges* of the hypergraph. Throughout this work, we only consider hypergraphs without multiple edges and thus, being E a usual set. If there is a risk of confusion we will denote the vertex set and the edge set of a hypergraph H explicitly by $V(H)$ and $E(H)$, respectively. A hypergraph $H = (V, E)$ is *simple* if no edge is contained in any other edge and $|e| \geq 2$ for all $e \in E$. A *loop* at $x \in V$ is an edge $\{x\} \in E$. We denote by $\mathcal{L}H$ the hypergraph obtained from H by adding a loop at each vertex. Conversely, $\mathcal{N}H$ denotes the hypergraph obtained by removing all loops from H.

Two vertices u and v are *adjacent* in $H = (V, E)$ if there is an edge $e \in E$ such that $u, v \in e$. The set of all vertices u that are adjacent to v in H is denoted by $N_H(v)$. The set $N_H[v] = N_H(v) \cup \{v\}$ is called the *(closed) neighborhood* of v. Unless there is a risk of confusion, we denote $N_H(v)$ and $N_H[v]$ simply with $N(v)$ and $N[v]$, respectively. If any two distinct vertices can be distinguished by their neighborhoods, that is, $N[u] \neq N[v]$ holds for all $u, v \in V$, then the hypergraph $H = (V, E)$ is called *thin*. If for two edges $e, f \in E$ holds

$e \cap f \neq \emptyset$, we say that e and f are *adjacent*. A vertex v and an edge e of H are *incident* if $v \in e$. The *degree* $\deg_H(v)$ of a vertex $v \in V$ is the number of edges incident to v in H. The *maximum degree* $\max_{v \in V} \deg_H(v)$ is denoted by $\Delta(H)$.

The *rank* of a hypergraph $H = (V, E)$ is $r(H) = \max_{e \in E} |e|$, the *anti-rank* is $s(H) = \min_{e \in E} |e|$. A *uniform hypergraph* H is a hypergraph such that $r(H) = s(H)$. A simple uniform hypergraph of rank r will be called *r-uniform*. A graph is thus a hypergraph s.t. $r(H) \leq 2$, a simple graph is a 2-uniform hypergraph.

Partial Hypergraphs. A *partial hypergraph* $H' = (V', E')$ of a hypergraph $H = (V, E)$, denoted by $H' \subseteq H$, is a hypergraph such that $V' \subseteq V$ and $E' \subseteq E$. In the class of graphs partial hypergraphs are called subgraphs. The partial hypergraph $H' = (V', E')$ is *induced* if $E' = \{e \in E \mid e \subseteq V'\}$. Induced hypergraphs will be denoted by $\langle V' \rangle$. H' is *generated* by E' if $V' = \bigcup_{e \in E'} e$. A partial hypergraph of a simple hypergraph is always simple. The *star* $S_H(v)$ with center $v \in V$ is the partial hypergraph of H generated by the edges containing v. With this terminology, we have $\deg_H(v) = |E(S_H(v))|$.

Walks, Paths, Distance. A *walk* in $H = (V, E)$ is a sequence $P_{v_0,v_k} = (v_0, e_1, v_1, e_2, \ldots, e_k, v_k)$, where $e_1, \ldots, e_k \in E$ and $v_0, \ldots, v_k \in V$, such that each $v_{i-1} \neq v_i$ and $v_{i-1}, v_i \in e_i$ for all $i = 1, \ldots, k$. The walk P_{v_0,v_k} is said to *join* the vertices v_0 and v_k. A *path* is a walk where both the vertices $v_0, \ldots, v_k$ and the edges $e_1, \ldots, e_k$ are all distinct. A path between two edges e_i and e_j is path P_{v_i,v_j} joining any pair of vertices $v_i \in e_i$ and $v_j \in e_j$. A *cycle* of length k, or k-cycle, is a sequence $(v_0, e_1, v_1, e_2, \ldots, v_{k-1}, e_k, v_0)$, such that $P_{v_0,v_{k-1}}$ is a path.

The *distance* $d_H(v, v')$ between two vertices v_0, v_k of H is the length of a shortest path joining them. We set $d_H(v, v') = \infty$ if there is no such path. A hypergraph $H = (V, E)$ is called *connected*, if any two vertices are joined by a path. A maximal connected partial hypergraph of H is called *connected component* of H. A partial hypergraph $H' \subseteq H$ is called *convex*, if all shortest paths in H between two vertices in H' are also contained in H'. We say that H' is *k-convex* if for any pair of vertices $u, v \in V(H')$ of distance $d_H(u, v) \leq k$, the set of all shortest paths from u to v in H is also contained in H'. For general H', i.e., if H' is not necessarily connected, we call H' (k-)convex, if all of its connected components are (k-)convex.

Homomorphisms. For two hypergraphs H, H' a *homomorphism* from H into H', denoted with $f : H \to H'$, is a mapping $f : V(H) \to V(H')$ such that $f(e) := \{f(v_1), \ldots, f(v_r)\}$ is an edge in H' if $e = \{v_1, \ldots, v_r\}$ is an edge in H. Note, a homomorphism from H into H' implies also a mapping $f_{\mathcal{E}} : E(H) \to E(H')$. A mapping $f : V(H) \to V(H')$ is a *weak homomorphism* if edges are mapped either on edges or on vertices.

A homomorphism f that is bijective is called an *isomorphism* if holds $f(e) \in E(H')$ if and only if $e \in E(H)$. We say, H and H' are *isomorphic*, in symbols $H \cong H'$, if there exists an isomorphism between them. An isomorphism from a hypergraph H onto itself is an *automorphism*.

Let $f : H \to H'$ be a weak homomorphism. By abuse of language, the partial hypergraph of H that is induced by the vertex set $\{x \in V(H) \mid f(x) = v\}$ for $v \in V(H')$ is denoted with $f^{-1}(v)$. The partial hypergraph of H induced by the vertex set $\{x \in V(H) \mid f(x) \in e\}$ for $e \in E(H')$ is denoted with $f^{-1}(e)$. Note, $f^{-1}(v)$ and $f^{-1}(e)$ actually refers to sets. However, it will be clear from the context what is meant.

Relations. We will consider equivalence relations R on the edge set $E(H)$ of a hypergraph H. In the style of Section 2.3, we will use the following terminology: For an equivalence class $\varphi \sqsubseteq R$, an edge e is called φ-edge if $e \in \varphi$. The partial hypergraph H_φ has vertex set $V(H)$ and edge set φ. The connected component of H_φ containing vertex $x \in V(H)$ is called *φ-layer through x*, denoted by H^x_φ. Analogously, the subgraphs $H_{\overline{\varphi}}$ and $H^x_{\overline{\varphi}}$ are defined. For a φ-layer H^x_φ and a vertex $y \in V(H)$ holds either $y \in V(H)$ and thus $H^x_\varphi = H^y_\varphi$ or $H^x_\varphi \cap H^y_\varphi = \emptyset$. The star with center u generated by all φ-edges incident to u is denoted by $S_{H_\varphi}(u)$, or $S_\varphi(u)$ for short.

An equivalence relation R on the edge set $E(H)$ of a hypergraph H that has equivalence classes $\varphi_i, i \in I$ is called *convex* if for any $K \subseteq I$ the partial hypergraph H_χ with $\chi = \bigcup_{i \in K} \varphi_i$ is convex. The *convex hull* $\mathcal{C}(R)$ of a relation R is the minimal convex equivalence relation on $E(H)$ that contains R. An equivalence class $\varphi \sqsubseteq R$ is called *k-convex* if H_φ is k-convex. We say R is *k-convex* if each equivalence class of R is k-convex. R is called *weakly k-convex* if at least one equivalence class of R is k-convex.

2-sections. The 2-section $[H]_2$ of a hypergraph $H = (V, E)$ is the graph (V, E') with $E' = \{\{x, y\} \subseteq V \mid x \neq y, \exists\, e \in E : \{x, y\} \subseteq e\}$, that is, two vertices are adjacent in $[H]_2$ if they belong to the same hyperedge in H. Thus, every hyperedge of H is a clique in $[H]_2$. Note, the 2-section $[H]_2$ of a hypergraph $H = (V, E)$ is only uniquely determined if H is *conformal*, that is, for every subset $W \subseteq V$ holds that if $\langle W \rangle$ is a clique in $[H]_2$ then $W \in E$.

A very useful property of the 2-section is the following:

Lemma 6.1 (Distance Formula). *Let $H = (V, E)$ be a hypergraph and $x, y \in V$. Then the distances between x and y in H and in $[H]_2$ are the same.*

Proof. Note, x and y are in different connected components of H if and only if x and y are in different connected components of $[H]_2$ and hence, $d_H(x, y) = d_{[H]_2}(x, y) = \infty$. Thus, w.l.o.g. assume H (and hence $[H]_2$) to be connected. Let $P = (x, e_1, v_1, \ldots, v_{k-1}, e_k, y)$ denote a shortest path between x and y in H. By construction of $[H]_2$ there is a walk $P' = (x, e'_1, v_1, \ldots, v_{k-1}, e'_k, y)$ in $[H]_2$. Thus, $k = d_H(x, y) \geq d_{[H]_2}(x, y) = l$. Assume, $k > l$. Then there is a path $Q' = (x, f'_1, v_1, \ldots, v_{k-1}, f'_l, y)$ in $[H]_2$. Thus, for all f'_i there is an edge $f_i \in E(H)$ such that $f'_i \subseteq f_i$ and hence, a walk of length l in H, a contradiction. $\square$

6.2 Hypergraph Products

As shown in [37], it is possible to find several non-equivalent generalizations of the standard graph products to hypergraph products. We define in the following the Cartesian product $\Box$, the normal product $\breve{\boxtimes}$ and the strong product $\widehat{\boxtimes}$, where the latter two products can be considered as generalizations of the usual strong *graph* product.

The Cartesian Product

Let H_1 and H_2 be two hypergraphs. The *Cartesian product* $H = H_1 \Box H_2$ has vertex set $V(H) = V(H_1) \times V(H_2)$ and edge set

$$\begin{aligned} E(H) = & \big\{\{x\} \times f : x \in V(H_1), f \in E(H_2)\big\} \\ & \cup \big\{e \times \{y\} : e \in E(H_1), y \in V(H_2)\big\}. \end{aligned}$$

Restricted to graphs, this definition coincides with the Cartesian graph product. Moreover, it was shown in [37]:

Lemma 6.2 ([37])**.** *Let H and H' be two hypergraphs. Then*

$$[H' \Box H'']_2 = [H']_2 \Box [H'']_2.$$

In Figure 6.1(c), a Cartesian product of two hypergraphs is shown.

The Cartesian product is associative and commutative, thus the Cartesian product of arbitrarily many hypergraphs is well defined. The Cartesian product is distributive w.r.t. the disjoint union of hypergraphs. The one-vertex graph K_1 serves as a unit, that is, it holds the *trivial* product representation $H \Box K_1 \cong H$ for all hypergraphs H. The Cartesian product of two hypergraphs is connected if and only if both factors are connected. It is simple if and only if both factors are simple [37].

The mapping $p_i : V(\Box_{i=1}^n H_i) \to V(H_i)$ defined by $p_i(v) = v_i$ for $v = (v_1, v_2, \dots, v_n)$ is called *projection* onto the i-th factor of H. It is a weak homomorphism for all $i = 1, \dots, n$. With this terminology we have, e is an edge in $\Box_{i=1}^n H_i$ iff $p_j(e) \in E(H_j)$ for exactly one $j \in \{1, \dots, n\}$ and $|p_k(e)| = 1$ for all $k \neq j$. The induced partial hypergraph H_i^w of H with vertex set $V(H_i^w) = \{v \in V(H) \mid p_j(v) = w_j, \text{ for all } j \neq i\}$ is called *H_i-layer through w*. It is isomorphic to H_i.

Prime Factor Decomposition. Analogously to graphs, a hypergraph H is called *prime* (w.r.t. Cartesian product) if it has only trivial product representation. Imrich [39] showed the following:

Theorem 6.3 ([39])**.** *Every connected hypergraph has unique representation as the Cartesian product of prime hypergraphs.*

Since graphs are special hypergraphs, PFD need not be unique for disconnected hypergraphs in general. Bretto et al. constructed an algorithm, that computes the PFD of a connected hypergraph in polynomial time [6].

An equivalence relation R on the edge set $E(H)$ of a Cartesian product $H = \Box_{i=1}^{n} H_i$ of (not necessarily prime) hypergraphs H_i is a *product relation* if $e \; R \; f$ holds if and only if there exists a $j \in \{1, \dots, n\}$ such that $|p_j(e)| > 1$ and $|p_j(f)| > 1$. It has been shown in [53], that the product relation σ according to the PFD of a hypergraph is just the convex hull of a certain relation δ, $\sigma = \mathcal{C}(\delta)$, that will be defined in Section 7.1.

The Normal Product and The Strong Product

As the Cartesian product, the *normal product* $H_1 \breve{\boxtimes} H_2$ and the *strong product* $H_1 \widehat{\boxtimes} H_2$ have vertex set $V(H_1 \breve{\boxtimes} H_2) = V(H_1 \widehat{\boxtimes} H_2) = V(H_1) \times V(H_2)$. A subset e of $V(H_1) \times V(H_2)$ is an edge in the normal product, $e \in E(H_1 \breve{\boxtimes} H_2)$, if and only if one of the following conditions is satisfied:

(i) $e \in E(H_1 \Box H_2)$, or

(ii$_\mathrm{n}$) $p_i(e) \subseteq e_i \in E(H_i)$, and $|e| = |p_i(e)| = \min_{j=1,2}\{|e_j|\}$, for $i = 1, 2$.

e is an edge in the strong product, $e \in E(H_1 \widehat{\boxtimes} H_2)$, if and only if one of the following conditions is satisfied:

(i) $e \in E(H_1 \Box H_2)$, or

(ii$_\mathrm{s}$) $p_i(e) \in E(H_i)$, for $i = 1, 2$, and $|e| = \max_{i=1,2}\{|p_i(e)|\}$

For other equivalent definitions, see [37]. Note, if H_1 and H_2 are simple graphs, then the normal and strong (hypergraph) product coincides with the usual strong *graph* product [28]. The edges, henceforth, of the normal and the strong product, fulfilling Condition (i) are called *Cartesian* edges w.r.t. the factorization $H_1 \boxtimes H_2$, and the other edges are called *non-Cartesian* w.r.t. $H_1 \boxtimes H_2$, $\boxtimes \in \{\breve{\boxtimes}, \widehat{\boxtimes}\}$, see also Figure 6.1.

For later reference we state the next lemma. Note, the set of non-Cartesian edges w.r.t. $H_1 \boxtimes H_2$ arise from $E(H_1 \boxtimes H_2) \setminus E(H_1 \Box H_2)$ for $\boxtimes \in \{\breve{\boxtimes}, \widehat{\boxtimes}\}$.

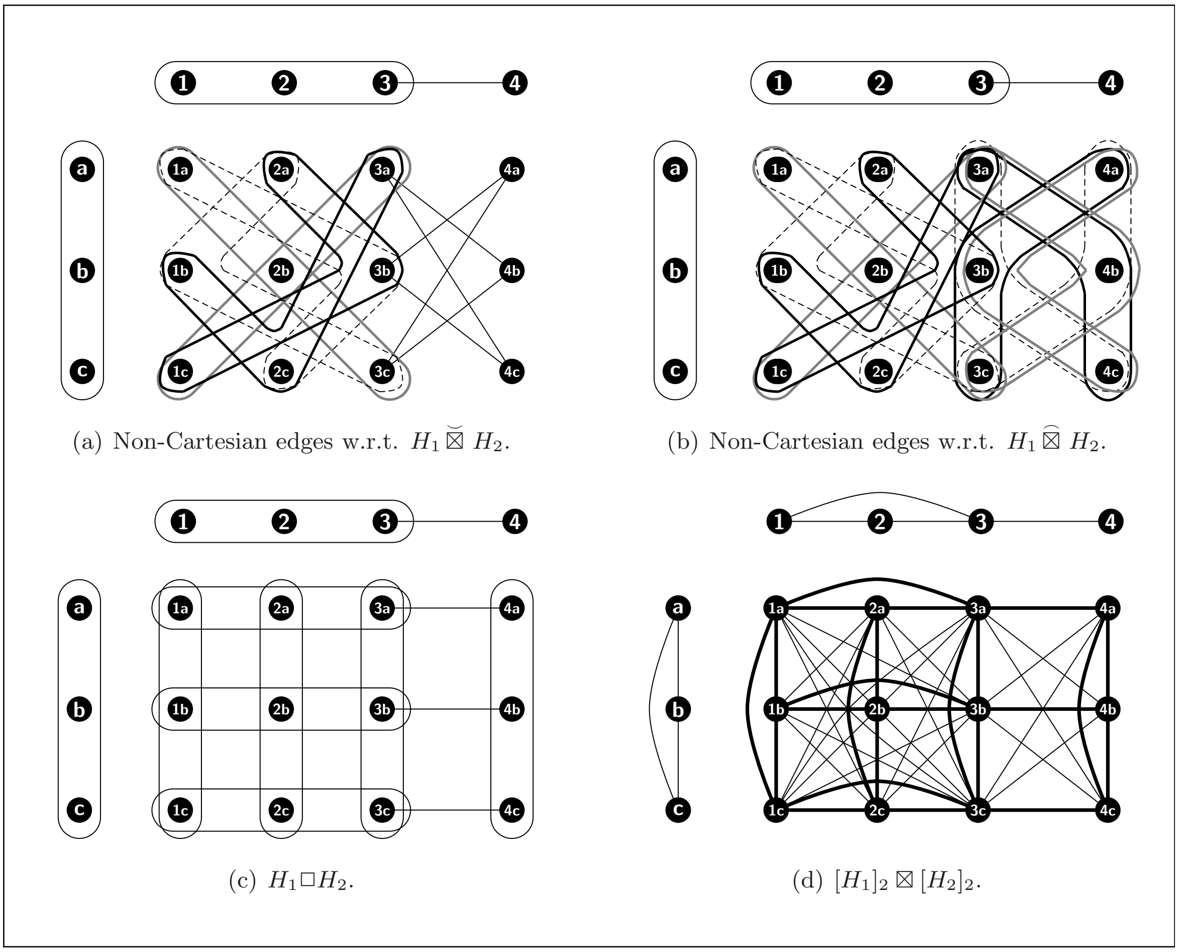
Lemma 6.4. *Let H_1, H_2 be two hypergraphs. For the number $|\breve{\times}|$ of non-Cartesian edges in $H = H_1 \breve{\boxtimes} H_2$ holds*

$$|\breve{\times}| = \sum_{e_1 \in E_1, e_2 \in E_2} \frac{(\max\{|e_1|, |e_2|\})!}{\big||e_1| - |e_2|\big|!}.$$

For the number $|\widehat{\times}|$ of non-Cartesian edges in $H = H_1 \widehat{\boxtimes} H_2$ holds

$$|\widehat{\times}| = \sum_{e_1 \in E_1, e_2 \in E_2} (\min\{|e_1|, |e_2|\})! S_{\max\{|e_1|,|e_2|\}, \min\{|e_1|,|e_2|\}},$$

where $S_{n,k}$ denotes the Stirling number of the second kind $S_{n,k} = \frac{1}{k!}\sum_{j=0}^{k} (-1)^{k-j} \binom{k}{j} j^n$.



(a) Non-Cartesian edges w.r.t. $H_1 \breve{\boxtimes} H_2$.

(b) Non-Cartesian edges w.r.t. $H_1 \widehat{\boxtimes} H_2$.

(c) $H_1 \square H_2$.

(d) $[H_1]_2 \boxtimes [H_2]_2$.

Fig. 6.1: Depicted are the non-Cartesian edges w.r.t. (a) the normal product $H_1 \breve{\boxtimes} H_2$ and (b) the strong product $H_1 \widehat{\boxtimes} H_2$, (c) the Cartesian edges of both products under investigation, and (c) the 2-section $[H_1]_2 \boxtimes [H_2]_2 = [H_1 \breve{\boxtimes} H_2]_2 = [H_1 \widehat{\boxtimes} H_2]_2$. The non-Cartesian edges are drawn in different line-styles, to improve visualization. The hypergraph factors H_1 and H_2 are not thin, and thus neither $H_1 \breve{\boxtimes} H_2$ nor $H_1 \widehat{\boxtimes} H_2$ is.

The basic idea of this proof is to show for all non-Cartesian edges $e \in E(H_1 \boxtimes H_2)$ that in the case of $|\breve{\times}|$, $p_i(x) \mapsto p_j(x)$ for all $x \in e$ defines an injective mapping $e_i \to e_j$, whenever $|e_i| \leq |e_j|$, and in the case of $|\widehat{\times}|$, $p_i(x) \mapsto p_j(x)$ for all $x \in e$ defines a surjective mapping $e_i \to e_j$, whenever $|e_i| \geq |e_j|$, where e is composed of $e_i \in E(H_i)$ and $e_j \in E(H_j)$, $i \neq j \in \{1, 2\}$. However, the proof is very long, therefore, we omit it here and put it in the appendix.

Remark 6.5. *For the normal product $H = H_1 \breve{\boxtimes} H_2$ and an edge $e \in E(H)$ holds, if $p_i(e) \subseteq e_i \in E(H_i)$ then $|e| \leq |e_i|$. In particular, $p_i(e) \subseteq e_i \in E(H_i)$ and $|e| = |e_i|$ implies that $p_i(e) = e_i \in E(H_i)$, $i \in \{1, 2\}$.*

For the strong product $H = H_1 \widehat{\boxtimes} H_2$ and an edge $e \in E(H)$ holds, if $p_i(e) = e_i \in E(H_i)$ then $|e| \geq |e_i|$. In particular, $p_i(e) = e_i \in E(H_i)$ and $|e| = |e_i|$ implies that $p_i(x) \neq p_i(y)$ for all $x, y \in e$ with $x \neq y$, $i \in \{1, 2\}$.

The normal product and the strong product are associative and commutative, thus the product of finitely many factors is well defined. The one-vertex hypergraph K_1 without edges serves as unit element for normal and strong product, i.e., $K_1 \boxtimes H \cong H$, for all H and $\boxtimes \in \{\breve{\boxtimes}, \widehat{\boxtimes}\}$. The normal and strong product of connected hypergraphs is always connected. While the projections from a strong product hypergraph onto its factors are weak homomorphisms, this need not be the case for the normal product. However, the projections from a normal product hypergraph onto its factors map adjacent vertices to adjacent vertices or collapse them to one vertex. Layers are defined analogously as in case of Cartesian product. They are isomorphic to the factors. The normal, resp. strong product is simple if and only if all of its factors are simple. The restriction of both products to the class of simple graphs coincides with the strong graph product [37].

Moreover, it was shown:

Lemma 6.6 ([37])**.** *Let H and H' be two hypergraph. Then it holds for $\boxtimes \in \{\breve{\boxtimes}, \widehat{\boxtimes}\}$*

$$[H' \boxtimes H'']_2 = [H']_2 \boxtimes [H'']_2.$$

As for the strong graph product $G = G' \boxtimes G''$ holds that G is thin if and only if G' and G'' are thin [28], we obtain together with the latter the following results.

Lemma 6.7. *Let $H = H' \boxtimes H''$, $\boxtimes \in \{\widehat{\boxtimes}, \breve{\boxtimes}\}$. Then it holds $N^H[x] = N^{[H]_2}[x]$. Moreover, H is thin if and only if $[H]_2$ is thin if and only if H' and H'' are thin.*

Remark 6.8. *In the sequel of this work, we will use the symbol $\boxtimes$ for both products, that is, $\boxtimes \in \{\breve{\boxtimes}, \widehat{\boxtimes}\}$, unless there is a risk of confusion.*

Furthermore, for sake of convenience, we introduce the following notations. Let H_1 and H_2 be hypergraphs and $\circledast \in \{\Box, \breve{\boxtimes}, \widehat{\boxtimes}\}$. For $H_1 \circledast H_2$ let $e_i \in E(H_i), i = 1, 2$ and define

$$e_1 \circledast e_2 := (e_1, \{e_1\}) \circledast (e_2, \{e_2\}).$$

Note, for $\circledast \in \{\Box, \breve{\boxtimes}, \widehat{\boxtimes}\}$ holds $E(e_1 \circledast e_2) \subseteq E(H_1 \circledast H_2)$. Moreover, for an arbitrary subset $E' \subseteq E(H_1)$ and $x \in V(H_2)$ we denote by $E' \times \{x\} := \{e \times \{x\} \mid e \in E'\}$. For later reference we remark, since K_1 is the unit element for $\circledast$ we can rewrite $E' \times \{x\} = E((V', E') \circledast (x, \emptyset))$ where $V' = \cup_{e \in E'} e$.

Chapter 7

The Grid Property and (Cartesian) Product-like Hypergraphs

Equivalence relations on the edge set of a hypergraph that satisfy the "grid property", a certain restrictive condition on diagonal-free grids, that can be seen as a generalization of the "square property" on graphs, introduced in Chapter 3, play a crucial role in the theory of Cartesian hypergraph products. In particular, it was shown in [53] that every convex equivalence relation with the grid property induces a factorization w.r.t. the Cartesian product. In the class of graphs, even non-convex relations with the square property provide rich structural information on local isomorphisms, local product structures, and product structures of quotient graphs. As we have seen in the first part of this thesis, even relaxations of the square property still retain (some of) these properties. Here, we examine the grid property and its relaxations in its own right. Vertex partitions derived from these equivalence classes of the edges give rise to equivalence relations on the vertex set. This in turn determines quotient graphs that have non-trivial product structures. Furthermore, as the (unique) square property in the graph case, the grid property can be expected to play an important role for Cartesian hypergraph bundles, the hypergraph analog of Cartesian graph bundles. We will explore this topic in detail in Section 7.3.

7.1 The Grid Property

We start by defining grids in hypergraphs. As in the case of cycles in graphs, which are conveniently defined as collections of edges, we regard them as collections of (hyper)edges.

Definition 7.1 (Grid). *An* $r \times s$-grid *in a hypergraph* $H = (V, E)$ *is a collection* $\mathcal{G} = \{e_1, \dots, e_s, f_1, \dots, f_r\} \subseteq E$ *of edges such that* *(i)* $|e_i \cap f_j| = 1$, *and*

(ii) $e_i \cap e_{i'} = f_j \cap f_{j'} = \emptyset$,

for all $i, i' \in \{1, \ldots, s\}$, $j, j' \in \{1, \ldots, r\}$, *with* $i \neq i'$, $j \neq j'$. *We say that* e_i *and* e_j *as well as* f_i *and* f_j *are* parallel edges *of* $\mathcal{G}$. *Each pair* e_i, f_j *of edges with* $i \in \{1, \ldots, s\}, j \in \{1, \ldots, r\}$ *is said to* span *the grid* $\mathcal{G}$.

A diagonal *in* $\mathcal{G}$ *is an edge* $d \in E(H)$ *satisfying*

$$e_k \cap f_l \cap d \neq \emptyset \quad \text{and} \quad e_{k'} \cap f_{l'} \cap d \neq \emptyset$$

for some $k, k' \in \{1, \ldots, s\}$ *and* $l, l' \in \{1, \ldots, r\}$ *with* $k \neq k'$ *and* $l \neq l'$.

The significance of *diagonal-free* grids is that they appear as the Cartesian product of two hyperedges. Thus, they can be seen as a natural generalization of the chordless squares that appear as products of edges in the Cartesian graph product.

In [53], the following generalization of the "square property" for equivalence relations on the edge set of simple hypergraphs was introduced:

Definition 7.2 (Grid Property)**.** *Let* R *be an equivalence relation on the edge set* $E(H)$ *of a hypergraph* H. *We say* R *has the* grid property *if*

(S1) Any two adjacent edges e *and* f *of* H *belonging to distinct* R*-equivalence classes span exactly one diagonal-free* $|e| \times |f|$*-grid* $\mathcal{G}$ *with parallel edges in the same equivalence class and*

(S2) Parallel edges in any diagonal-free grid $\mathcal{G}$ *of* $E(H)$ *are in the same* R*-equivalence class.*

The restriction of these statements to simple graphs recovers Definition 3.2 of the *square property.* In graphs, an equivalence relation with the square property is readily constructed as the transitive closure of the relation δ, see Section 3.1. In [53] the following generalization to hypergraphs has been introduced:

Definition 7.3 (Relation δ)**.** *Let* H *be a connected hypergraph. Two edges* $e, f \in E(H)$ *are* in relation δ, $e\ \delta\ f$, *if one of the following conditions holds:*

(i) $e = f$, *or*

(ii) $e \cap f = \emptyset$ *and* e *and* f *are opposite edges of a four-cycle, or*

(iii) $e \cap f \neq \emptyset$ *and there is no* $(|e| \times |f|)$*-grid without diagonals containing them.*

The relation δ is reflexive and symmetric. Its transitive closure δ^* is therefore an equivalence relation. As shown in [53], δ^* has the grid property and any equivalence relation that contains δ has the grid property. Moreover, we can state the following

Lemma 7.4. *If* R *is an equivalence relation on* $E(H)$ *satisfying the grid property and* S *is a coarser equivalence relation,* $R \subseteq S$, *then* S *also has the grid property.*

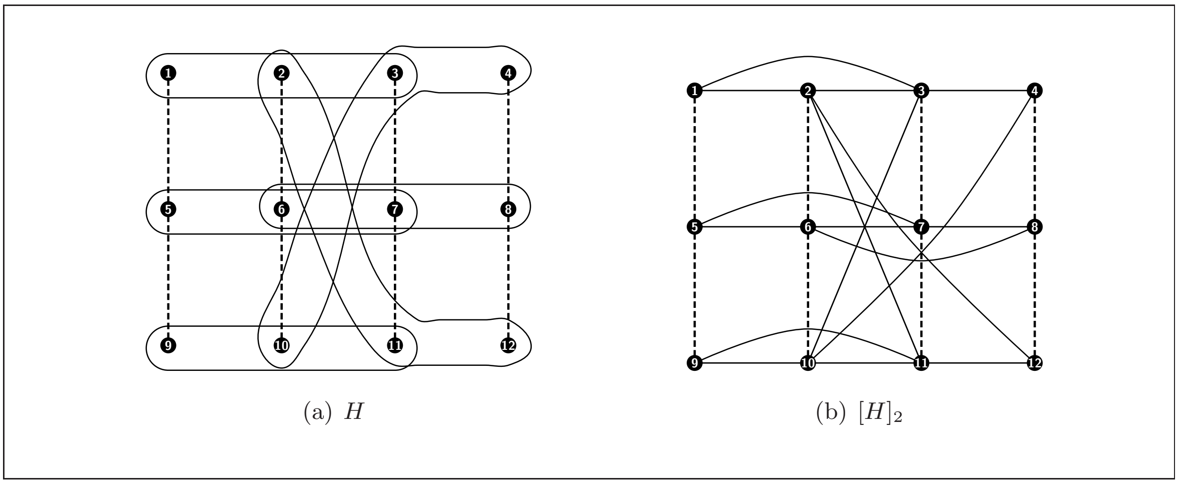


Fig. 7.1: **(a)** A hypergraph H with an equivalence relation R on $E(H)$ that consists of two equivalence classes indicated by edges with $|e| = 2$ (dashed) and edges with $|e| = 3$ (solid ovals). R has the grid property, but $\delta \not\subseteq R$, since, e.g., $(\{2,11,12\},\{3,7\}) \in \delta$ but $(\{2,11,12\},\{3,7\}) \notin R$. **(b)** The 2-section $[H]_2$ together with the equivalence relation R_2 on $E([H]_2)$ induced by R. Its equivalence classes are depicted by dashed and solid edges, respectively. R_2 does not satisfy the square property, since edges $\{2,3\}$ and $\{2,6\}$ span more than one square.

Conversely, if a relation R on $E(H)$ satisfies the grid property, this does not necessarily imply $\delta \subseteq R$, as shown in Fig. 7.1. However, we can formulate more restrictive conditions in terms of the 2-section $[H]_2$ of the hypergraph H. Let R be an equivalence relation on $E(H)$. Then R induces a relation R_2 on $E([H]_2)$ by setting $e' \; R_2 \; f'$ for $e', f' \in E([H]_2)$ iff there are edges $e, f \in E(H)$ with $e \; R \; f$ and $e' \subseteq e$, $f' \subseteq f$.

Lemma 7.5. *If R has the grid property then R_2 is an equivalence relation on $E([H]_2)$.*

Proof. Since R_2 is clearly reflexive and symmetric, we only need to show that R_2 is transitive. Therefore, let $e', f', g' \in E([H]_2)$ and suppose $e' \; R_2 \; f'$ and $f' \; R_2 \; g'$. By construction, there are $e, f, \hat{f}, g \in E(H)$ such that $e' \subseteq e, f' \subseteq f, f' \subseteq \hat{f}, g' \subseteq g$ and $e \; R \; f$ as well as $\hat{f} \; R \; g$. Furthermore, $f' \subseteq f \cap \hat{f}$, thus, $|f \cap \hat{f}| \geq 2$, which implies $f \; R \; \hat{f}$ because R satisfies the grid property. Since R is an equivalence relation, we can conclude $e \; R \; g$ and therefore also $e' \; R_2 \; g'$. □

The induced relation R_2 need not have the square property, although R has the grid property. However, it has the relaxed square property. This leads to the following definition.

Definition 7.6 (Strong Grid Property)**.** *An equivalence relation R on $E(H)$ has the* strong grid property *if R has the grid property and the induced equivalence relation R_2 on $E([H]_2)$ has the square property.*

An alternative characterization of the strong grid property is given by the following:

Proposition 7.7. *An equivalence relation R on $E(H)$ has the strong grid property if and only if $\delta \subseteq R$.*

Proof. First, let $\delta \subseteq R$. Then R has the grid property. We have to show that R_2 on $E([H_2])$ satisfies the square property. Therefore let $e' = \{x, y\}, f' = \{y, z\} \in E([H]_2)$ be two adjacent edges such that $(e', f') \notin R_2$. Then there exists adjacent edges $e, f \in E(H)$ with $e' \subseteq e$, $f' \subseteq f$ such that $(e, f) \notin R$. Thus, e and f span a unique diagonal-free grid $\mathcal{G} = \{e, e_1, \ldots, e_k, f, f_1, \ldots, f_l\}$ with $|e| = l + 1$ and $|f| = k + 1$. W.l.o.g., let $e \cap f_1 = \{x\}$ and $e_1 \cap f = \{z\}$. Hence, e' and f' span a square (x, y, z, w) with $\{w\} = e_1 \cap f_1$. This square must be chordless, since for any chord d' there exists a diagonal $d \supseteq d'$ of the grid $\mathcal{G}$.

Suppose there exists another square (x, y, z, v) spanned by e' and f'. Hence, there exist $\hat{e}, \hat{f}$ in $E(H)$ such that $\{x, v\} \subseteq \hat{e}$ and $\{v, z\} \subseteq \hat{f}$. Then neither $\hat{e}$ nor $\hat{f}$ are contained in $\mathcal{G}$, since otherwise $\hat{f}$ or $\hat{e}$, respectively, would be a diagonal of $\mathcal{G}$. If $\hat{e} = \hat{f}$ holds, then $\hat{e}$ would be a diagonal of this grid. Hence, $\hat{e} \neq \hat{f}$ must hold. However, this implies $f_1 \; \delta \; \hat{f}$ as well as $e \; \delta \; \hat{f}$ and therefore $e \; \delta^* \; f$ and consequently $e \; R \; f$, a contradiction. Thus, R has the strong grid property if $\delta \subseteq R$.

Now, assume R has the strong grid property. We have to show that for any two edges $e, f \in E(H)$ with $e \; \delta \; f$ holds $e \; R \; f$. First, suppose $e \; \delta \; f$ such that e and f are not adjacent. Thus, e and f must be opposite edges of a 4-cycle. Hence, there exists $e', f' \in E([H]_2)$ with $e' \subseteq e$ and $f' \subseteq f$ such that e' and f' are opposite edges of a square in $[H]_2$. Since R_2 has the square property, we can conclude $e' \; R_2 \; f'$. That is, there exists edges $\hat{e}, \hat{f} \in E(H)$ with $e' \subseteq \hat{e}$ and $f' \subseteq \hat{f}$ such that $\hat{e} \; R \; \hat{f}$. From $|f \cap \hat{f}| \geq 2$ and $|e \cap \hat{e}| \geq 2$, we can conclude $e \; R \; \hat{e}$ and $f \; R \; \hat{f}$ and finally $e \; R \; f$. Now, let $e, f \in E(H)$ be adjacent and suppose, for contraposition, $(e, f) \notin R$. Then e and f span a unique, diagonal-free grid. The definition of δ implies $(e, f) \notin \delta$. □

Instead of δ, we can also construct a less restrictive, i.e., finer, equivalence relation on $E(H)$ with the grid property.

Definition 7.8 (Relation γ)**.** *Let H be a connected hypergraph. Two edges $e, f \in E(H)$ are in relation γ, $e \; \gamma \; f$, if one of the following conditions holds:*

(i) $e = f$, *or*

(ii) $e \cap f = \emptyset$ *and e and f are parallel edges in a grid $\mathcal{G}$ in H, or*

(iii) $e \cap f \neq \emptyset$ *and there is no diagonal-free $(|e| \times |f|)$-grid that contains e and f, or*

(iv) $e \cap f \neq \emptyset$ *and there are at least two $(|e| \times |f|)$-grids containing e and f.*

By construction, γ is reflexive and symmetric. Its transitive closure γ^* is therefore an equivalence relation.

Proposition 7.9. *An equivalence relation R on $E(H)$ has the grid property if and only if $\gamma \subseteq R$.*

Proof. First, suppose $\gamma \subseteq R$. We have to show that R satisfies the grid property. Therefore, let $e, f \in E(H)$ be two adjacent edges such that $(e, f) \notin R$, hence, $(e, f) \notin \gamma$. Then from condition *(iii)* in the definition of γ, we can conclude that there exists a grid $\mathcal{G}$ in H spanned by e and f, and from condition *(iv)* it follows that this grid is unique.

Now, assume R has the grid property. We have to show that $e\ R\ f$ for all $e, f \in E(H)$ with $e\ \gamma\ f$. First, let $e\ \gamma\ f$ s.t. e and f are not adjacent. Thus, e and f must be parallel edges in a grid, and therefore $e\ R\ f$. Now, let e and f be adjacent and suppose, for contraposition, $(e, f) \notin R$. Then e and f span a unique grid without diagonals. From the definition of γ we can conclude $(e, f) \notin \gamma$ since neither *(iii)* nor *(iv)* is fulfilled. ☐

If H is a graph, then we do not have to insist on condition *(iv)* in the definition of γ. To see this, consider the relation γ_0 defined by the conditions *(i)*, *(ii)*, and *(iii)* in Definition 7.8. If e and f span more than one square in a graph, then they are guaranteed to be in the transitive closure γ_0^* of γ_0, see Fig. 7.2. This is not true in a hypergraph, however, as the example in Fig. 7.2 shows.

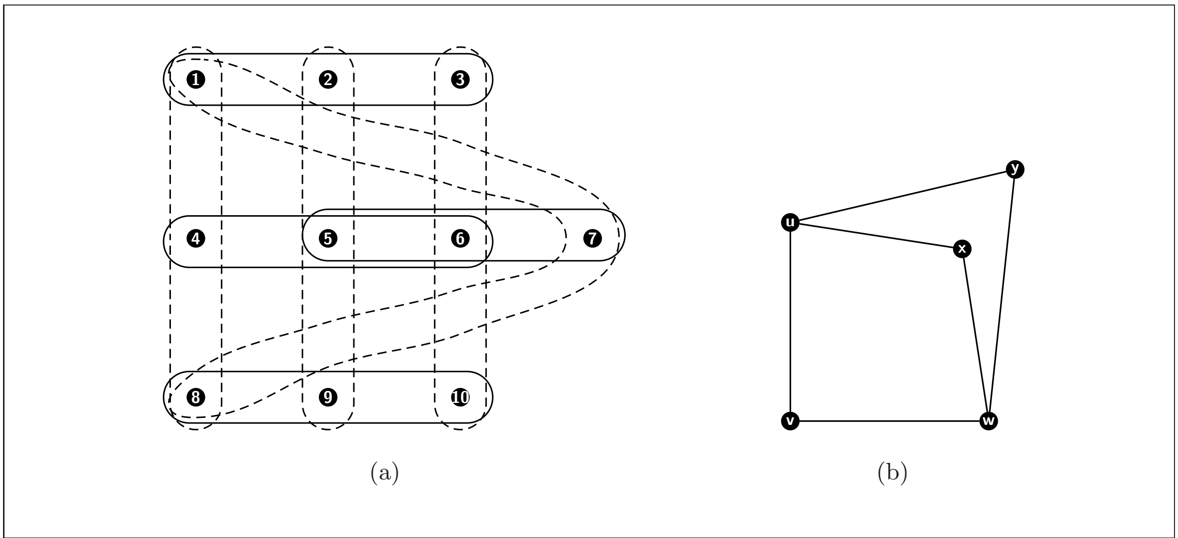


Fig. 7.2: **(a)** The equivalence relation $\gamma_0 = \gamma_0^*$ on $E(H)$, whose classes are indicated by dashed and solid ovals, resp., satisfies one of the conditions (i), (ii), and (iii) of Definition 7.8, but violated condition (iv) since there are two grids spanned by edges $\{1, 2, 3\}$ and $\{2, 5, 9\}$. **(b)** The edges $\{u, v\}$ and $\{v, w\}$ in the graph span two squares, $u - v - w - x$ and $u - v - w - y$. It holds: $\{u, v\}\gamma_0\{w, y\}$, $\{w, y\}\gamma_0\{u, x\}$ and $\{u, x\}\gamma_0\{v, w\}$, hence $\{u, v\}\gamma_0^*\{v, w\}$.

Corollary 7.10. *For any connected hypergraph H holds $\gamma_0^* \subseteq \gamma^* \subseteq \delta^*$. If H is a graph, then $\gamma_0^* = \gamma^* = \delta^*$.*

As it turns out, for our purpose it will suffice to consider equivalence relations of $E(H)$ that satisfy a much less restrictive condition than the grid property:

Definition 7.11 (Relaxed Grid Property)**.** *Let R be an equivalence relation on the edge set $E(H)$ of a connected hypergraph H. We say R has the* relaxed grid property *if any two adjacent edges e, f of H that belong to distinct equivalence classes of R span an $|e| \times |f|$-grid $\mathcal{G}$ with parallel edges in the same equivalence class of R.*

Restricted to graphs, this definition coincides with the relaxed square property, see Definition 3.4. One easily verifies the follow analog of Prop. 7.4:

Lemma 7.12. *If R is an equivalence relation on $E(H)$ satisfying the relaxed grid property and S is a coarser equivalence relation, $R \subseteq S$, then S also has the relaxed grid property.*

As in the graph case, see Section 3.2, there is in general no unique finest equivalence relation with the relaxed grid property, in contrast to the grid property.

In the remainder of this section we collect several useful properties of the (relaxed) grid property that generalize well known results for the (relaxed) square property on the class of graphs. Some of these results have been shown in previous work, others are novel to our knowledge.

Lemma 7.13. *Let R be an equivalence relation on $E(H)$ with the relaxed grid property. Then each vertex of H is incident to at least one edge of each R-class.*

Lemma 7.14. *Let R be an equivalence relation on $E(H)$ with the relaxed grid property that has only two equivalence classes φ and $\overline{\varphi}$. Then $|V(H^x_\varphi) \cap V(H^y_{\overline{\varphi}})| \geq 1$ for all $x, y \in V(H)$.*

If R is convex and has the grid property, then $|V(H^x_\varphi) \cap V(H^y_{\overline{\varphi}})| = 1$ holds for all $x, y \in V(H)$ and all $\varphi \sqsubseteq R$ [53]. As in the graph case, one argues, that any convex equivalence relation satisfying the relaxed grid property already has the grid property. Consequently, this result applies to equivalence relations with relaxed grid property as well.

Lemma 7.13 and Lemma 7.14 have been proved in [53] assuming the grid property. The proofs, however, use the relaxed grid property only. Since the proofs are quite similar to those of Lemma 3.11, resp. Lemma 3.13 in the graph case, we omit them here.

In the class of graphs, it is known that equivalence relations on the edge set of a connected graph that have the square property induce certain local isomorphisms [15]. This generalizes also to the unique square property [66] and further to the well-behaved RSP-relations, see Section 3.2. As we have noticed ibidem, a corresponding result does not necessarily hold for equivalence relations satisfying only the relaxed square property, however. Even equivalence relations with the grid property do not necessarily induce this kind of local isomorphisms. But we have the following:

Lemma 7.15. *Let R be an equivalence relation on $E(H)$ with the grid property, $\varphi, \psi \sqsubseteq R$, $\psi \neq \varphi$, $e \in \varphi$, and $x, y \in e$. Then $m(S_\psi(x)) = m(S_\psi(y))$.*

Proof. W.l.o.g., suppose $E(S_\psi(x)) = \{f_1, \ldots, f_k\}$. The grid property implies that each f_i together with e spans a unique diagonal-free grid $\mathcal{G}_i$, $i = 1, \ldots, k$ and $\mathcal{G}_i \neq \mathcal{G}_j$ if $i \neq j$. Furthermore, for each $i = 1, \ldots, k$ there is an edge $f'_i \in \mathcal{G}_i$ such that $y \in f'_i$. Thus $\mathcal{G}_i$ is also spanned by f'_i and e. If $f'_i = f'_j$, this immediately implies $i = j$, otherwise, f'_i and e would span more than one grid. Therefore, we have $m(S_\psi(x)) \leq m(S_\psi(y))$. An analogous argument established and $m(S_\psi(y)) \leq m(S_\psi(x))$, hence equality must hold. $\square$

If we assume the strong grid property, we again find local isomorphism analogously to the graph case.

Lemma 7.16. *Let R be an equivalence relation on $E(H)$ with the strong grid property, $\varphi, \psi \sqsubseteq R$, $\psi \neq \varphi$, $e \in \varphi$, and $x, y \in e$. Then the stars generated by all ψ-edges centered in x and y, resp., are isomorphic, $S_\psi(x) \cong S_\psi(y)$.*

Proof. Let $e \in \varphi \subseteq E(H)$, $x, y \in V(H)$ such that $x, y \in e$. Let $E(S_\psi(x)) = \{f^1, \ldots, f^m\}$ and $E(S_\psi(y)) = \{g^1, \ldots g^{m'}\}$. We have $m = m'$ as a consequence of Lemma 7.15.

By the grid property, $e \in \varphi$ and $f^i \in \psi$ span a unique grid $\mathcal{G}^i = \{f^i, f^i_1, \ldots, f^i_l, e, e^i_1, \ldots, e^i_{k_i}\}$ in H for all $i = 1, \ldots, m$, with $|f^i| = k_i + 1$, $|e| = l + 1$. From the proof of Lemma 7.15, we can conclude that for all $i = 1, \ldots, m$ there exists a uniquely determined edge $g^j \in E(S_\psi(y))$ with $g^j \in \mathcal{G}_i$. W.l.o.g., let $g^i := f^i_1$ for all $i = 1, \ldots, m$. Moreover, set $e^i_0 := e$ and $f^i_0 := f^i$ for all $i = 1, \ldots, m$. By the definition of a grid, we have $e = \bigcup_{s=0}^{l}(f^i_s \cap e) = \bigcup_{s=0}^{l}(f^j_s \cap e)$ for all $i, j = 1, \ldots, m$. W.l.o.g., let $f^i_s \cap e = f^j_s \cap e$ for all $s = 0, \ldots, l$ and $i, j = 1, \ldots, m$.

The vertex set of $S_\psi(x)$ is $V(S_\psi(x)) = \bigcup_{i=1}^{m} f^i = \bigcup_{i=1}^{m}\bigcup_{r=1}^{k_i}(f^i \cap e^i_r)$. It will be convenient to relabel them in the following manner: We set $V(S_\psi(x)) \ni v := x^i_{r_i}$ iff v is the uniquely determined vertex with $\{v\} = f^i \cap e^i_{r_i}$. Note that vertices with different labels are not necessarily distinct. Analogously, we assign labels to vertices w of H as follows: $w := y^i_{r_i}$ iff w is the uniquely determined vertex with $\{w\} = g^i \cap e^i_{r_i}$. Since $y \in g^i$ for all $i = 1, \ldots, m$, it follows $y^i_{r_i} \in V(S_\psi(y))$ for all $r_i = 0, \ldots, k_i$, $i = 1, \ldots, m$.

With this notation, we have to prove that the map defined by

$$x^i_{r_i} \mapsto y^i_{r_i} \tag{7.1}$$

for all $r_i = 0, \ldots, k_i$, $i = 1, \ldots, m$, is an isomorphism between $S_\psi(x)$and $S_\psi(y)$. Thus, we have to show that $x^i_{r_i} = x^j_{s_j}$ if and only if $y^i_{r_i} = y^j_{s_j}$. If $i = j$ this immediately implies $r_i = s_i$, otherwise $e^i_{r_i} \cap e^i_{s_i} \neq \emptyset$. Now assume that $i \neq j$. Suppose first $x^i_{r_i} = x^j_{s_j}$, i.e., $f^i \cap e^i_{r_i} = f^j \cap e^j_{s_j}$. If $y^i_{r_i} \neq y^j_{s_j}$, then $g^i \cap e^i_{r_i} \neq g^j \cap e^j_{s_j}$, which implies $e^i_{r_i} \neq e^j_{s_j}$ because otherwise g^j would be a diagonal of the grid $\mathcal{G}_i$. But then we find a 4-cycle $(x^i_{r_i}, e^i_{r_i}, y^i_{r_i}, g^i, y, g^j, y^j_{s_j}, e^j_{r_j})$ in H, with $g^i, g^j \in \psi$ and $e^i_{r_i}, e^j_{s_j} \in \varphi$, which contradicts the strong grid property. Thus, $x^i_{r_i} = x^j_{s_j}$ implies $y^i_{r_i} = y^j_{s_j}$, i.e., the mapping $x^i_{r_i} \mapsto y^i_{r_i}$ is well defined.

From analogous considerations we can conclude $x^i_{r_i} = x^j_{s_j}$ if $y^i_{r_i} = y^j_{s_j}$, which proves injectivity. Furthermore, $V(S_\psi(y)) = \bigcup_{i=1}^{m} g^i = \bigcup_{i=1}^{m}\bigcup_{r=1}^{k_i}(g^i \cap e^i_r) = \bigcup_{i=1}^{m}\bigcup_{r=1}^{k_i}\{y^i_r\}$. Thus, this mapping is surjective and therefore bijective. Moreover, since the edges of $S_\psi(x)$ and $S_\psi(y)$ are given by $f^i = \bigcup_{r=0}^{l}(f^i \cap e^i_r) = \bigcup_{r=0}^{l}\{x^i_r\}$ and $g^i = \bigcup_{r=0}^{l}(g^i \cap e^i_r) = \bigcup_{r=0}^{l}\{y^i_r\}$, resp., this mapping is an isomorphism. □

The isomorphism $S_\psi(x) \cong S_\psi(y)$ given in Equation (7.1) is *induced* by the edge $e \in \varphi$ in the sense that vertices of $S_\psi(x)$ are mapped onto vertices of $S_\psi(y)$ if and only if they are in the same edge that is parallel to e. The grid property is by itself not sufficient to determine these local isomorphism, as Fig. 7.1 shows: The vertices 2 and 6 are connected by a dashed edge, but the stars generated by the solid edges centered in 2 and 6, respectively, are not isomorphic.

The following Lemma displays the connection between relations with the relaxed grid property on the edge set of a hypergraph and its 2-section and can be seen as a generalization of Lemma 6.2.

Lemma 7.17. *Let H be a hypergraph and let R be an equivalence relation on $E(H)$ that satisfies the relaxed grid property. It holds:*

(1) R_2 is an RSP-relation on $E([H]_2)$.

(2) $[H_\varphi]_2 = ([H]_2)_{\varphi'}$ for each equivalence class $\varphi \sqsubseteq R$, where φ' denotes the equivalence class of R_2 that is induced by φ such that $e' \in \varphi'$ whenever $E([H_2]) \ni e' \subseteq e \in \varphi$.

(3) $[H^x_\varphi]_2 = ([H]_2)^x_{\varphi'}$ for each equivalence class $\varphi \sqsubseteq R$ and all $x \in V(H)$.

Proof. (1) Since $|e \cap f| \geq 2$ implies $(e, f) \in R$ if R has the relaxed grid property, it is shown analogously as for Lemma 7.5 that R_2 is an equivalence relation. it remains to show that R_2 has the relaxed square property. Let $[x, y], [y, z]$ be two edges in $[H]_2$ such that $([x, y], [y, z]) \notin R_2$. By construction, there exists $e, f \in E(H)$ with $x, y \in e$ and $y, z \in f$ such that $(e, f) \notin R$. Since R has the relaxed grid property, there are edges $e', f' \in E(H)$ with $(e, e'), (f, f') \in R$ such that $e \cap f' = \{x\}$, $e' \cap f = \{z\}$ and $e' \cap f' = \{w\}$. Hence, $x - y - z - w$ forms a square in $[H]_2$ containing $[x, y]$ and $[y, z]$, that has opposite edges in the same equivalence classes.

(2) Clearly, we have $V([H_\varphi]_2) = V(H) = V\left(([H]_2)_{\varphi'}\right)$. It remains to show $E([H_\varphi]_2) = E\left(([H]_2)_{\varphi'}\right)$. Let $e' \in E\left(([H]_2)_{\varphi'}\right)$ which is equivalent to $e' \in \varphi'$. By construction, this is if and only if there exists $e \in \varphi$ with $e' \subseteq e$ which is equivalent to $e' \in E([H_\varphi]_2)$.

(3) Follows from (2) and Lemma 6.1 □

7.2 Quotient Hypergraphs

In this section, we prove that several results that have been shown for graphs in Chapter 5 are also true for hypergraphs and the relaxed grid property.

Definition 7.18 (Quotient Hypergraph). *Let $H = (V, E)$ be a hypergraph and let $\mathcal{P} = \{V_1, \dots, V_k\}$ be a partition of the vertex set V of H. The* quotient hypergraph *$H/\mathcal{P}$ has vertex set $V(H/\mathcal{P}) = \{V_1, \dots, V_k\}$ and $f = \{V_{i_1}, \dots, V_{i_r}\} \subseteq V(H/\mathcal{P})$ is an edge in $H/\mathcal{P}$ iff there exists an edge $e \in E(H)$ such that*

(i) $e \cap V_{i_j} \neq \emptyset$ for all $j = 1, \dots, r$ and

(ii) $e \subseteq \bigcup_{j=1}^{r} V_{i_j}$.

For later use, we record the following lemma.

Lemma 7.19. *Let H be a hypergraph. It holds*

$$[H/\mathcal{P}]_2 = [H]_2/\mathcal{P}$$

for any partition $\mathcal{P}$ of $V(H)$.

Proof. Since $V(H) = V([H]_2)$, by definition of the 2-section of a hypergraph, a partition on $V(H)$ is the same as a partition on $V([H]_2)$. Therefore, we have to show that any two classes $A, B \in \mathcal{P}$ are adjacent in $[H/\mathcal{P}]_2$ if and only they are adjacent in $[H]_2/\mathcal{P}$.

It holds $[A, B] \in E([H/\mathcal{P}]_2)$ if and only if there exists $a \in A$, $b \in B$ with $[a, b] \in E([H]_2)$. This is equivalent to that there exists an edge $e \in H$ with $a, b \in e$ and thus $e \cap A \neq \emptyset \neq e \cap B$, which in turn is equivalent to the existence of an edge f in $H/\mathcal{P}$ with $A, B \in f$. This is satisfied if and only if $[A, B] \in E([H/\mathcal{P}]_2)$. □

Let R be an equivalence relation on the edge set of a hypergraph H. By construction, the set

$$\mathcal{P}^R_\varphi := \left\{V(H^x_\varphi) \mid x \in V(H)\right\}$$

is a partition of $V(H)$ for every $\varphi \sqsubseteq R$. The quotient hypergraph $H/\mathcal{P}^R_\varphi$ has as its vertex set the φ-layers H^x_φ. The set $\{H^{x_1}_\varphi, \dots, H^{x_k}_\varphi\}$ is an edge iff there are edges $e \in E(H)$ such that $e \cap V(H^w_\varphi) \neq \emptyset$ if and only if $w \in V(H^{x_i}_\varphi)$ for $i \in \{1, \dots, k\}$.

In the following, we will be interested in particular in the complements of R-classes, i.e., in $\overline{\varphi} := E \setminus \varphi$. The corresponding partial hypergraphs are denoted by $H_{\overline{\varphi}}$, with layer $H^x_{\overline{\varphi}}$ for a given $x \in V(H)$. We observe that $y \in V(H^x_{\overline{\varphi}})$ if and only if there is a path $P := (x = x_0, e_1, x_1, \dots e_k, x_k = y)$ from x to y such that $e_i \notin \varphi$ for all $1 \leq i \leq k$.

Just like $\mathcal{P}^R_\varphi$, the sets

$$\mathcal{P}^R_{\overline{\varphi}} := \left\{V(H^x_{\overline{\varphi}}) \mid x \in V(H)\right\}$$

form a partition of $V(H)$ for every $\varphi \sqsubseteq R$. To see this, we note that $x \in V(H^x_{\overline{\varphi}})$ holds for all $x \in V(H)$. Thus, $P \neq \emptyset$ for all $P \in \mathcal{P}^R_{\overline{\varphi}}$ and $\bigcup_{P \in \mathcal{P}^R_{\overline{\varphi}}} P = V(H)$. Furthermore, $V(H^x_{\overline{\varphi}}) \cap V(H^y_{\overline{\varphi}}) \neq \emptyset$ if and only if x and y are in the same $\overline{\varphi}$-layer, i.e., if and only if $V(H^x_{\overline{\varphi}}) = V(H^y_{\overline{\varphi}})$.

We furthermore will need the intersections

$$V_R(x) := \bigcap_{\varphi \sqsubseteq R} V(H^x_{\overline{\varphi}})\,.$$

These sets form the classes of the common refinement of the partitions $\mathcal{P}^R_{\overline{\varphi}}$, $\varphi \sqsubseteq R$, i.e.,

$$\mathcal{P}^R := \left\{ \bigcap_{\varphi \sqsubseteq R} V(H_{\overline{\varphi}}(x)) \mid x \in V(H) \right\} = \{V_R(x) \mid x \in V(H)\}$$

is again a partition of $V(H)$.

The main statement of this section is the following factorization theorem for the quotient hypergraph $H/\mathcal{P}^R$. It generalizes Theorem 5.1 for simple graphs.

Theorem 7.20. *If R is an equivalence relation with the relaxed grid property on $E(H)$ then*

$$H/\mathcal{P}^R \cong \square_{\varphi \sqsubseteq R} H_\varphi/\mathcal{P}^R_\varphi.$$

To prove Theorem 7.20, we first have to establish

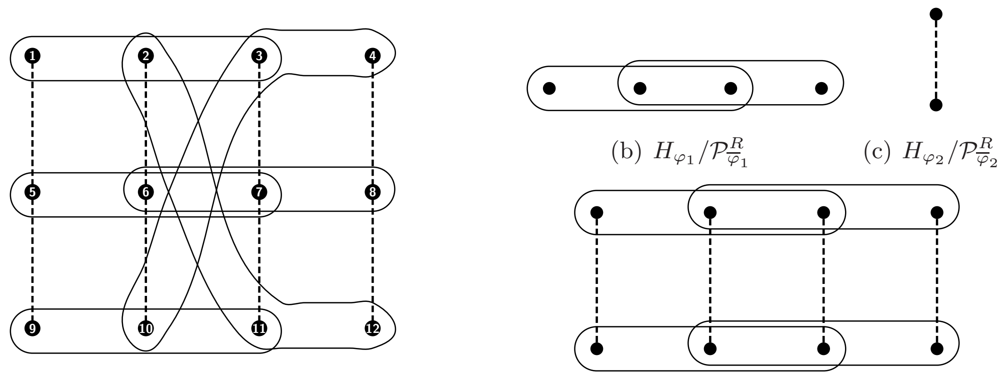
Lemma 7.21. *Let R be an equivalence relation on the edge set of a connected hypergraph H that satisfies the relaxed grid property and let $\varphi \sqsubseteq R$, $v \in V(H)$. Then for all $x \in V(H^v_\varphi)$ and all edges $e \in \psi \neq \varphi$ with $v \in e$, $\psi \sqsubseteq R$, there exists an edge e_x with $x \in e_x$ such that holds*

$$e \cap V(H^w_\varphi) = \emptyset \quad \textit{if and only if} \quad e_x \cap V(H^w_\varphi) = \emptyset$$

for all $w \in V(H)$.

Proof. Let $v \in e := e_0 \in \psi$. For any $x \in V(H^v_\varphi)$ there exists a path $P_{vx} := (v = v_0, f_1, v_1, \ldots, v_{k-1}, f_k, v_k = x)$ such that $f_i \in \varphi$ for all $i = 1, \ldots, k$. By the relaxed grid property, e_0 and f_1 span an $|e_0| \times |f_1|$-grid $\mathcal{G}_1$. Thus, there exists an edge $e_1 \in \mathcal{G}_1$, with $|e_1| = |e_0|$ such that $v_1 \in e_1 \in \psi$. Furthermore, for all $w \in e_0$ there is an edge $f^w_1 \in \varphi$ with $f^w_1 \cap e_1 \neq \emptyset$ if and only if $w \in f^w_1$. Hence, $e_0 \cap V(H^w_\varphi) = \emptyset$ if and only if $e_1 \cap V(H^w_\varphi) = \emptyset$. Moreover, since $v_1 \in f_2 \cap e_1$, $f_2 \in \varphi$, $e_1 \in \psi$, these two edges again span an $|e_1| \times |f_2|$-grid $\mathcal{G}_2$.

Inductively, we construct a collection of edges $e_1, \ldots, e_k$ such that e_{i-1} and f_i span an $|e_{i-1}| \times |f_i|$-grid $\mathcal{G}_i$ such that $e_i \in \mathcal{G}_i$ and $v_i \in e_i \cap f_i$. Therefore, by the same argument as before, we have $e_{i-1} \cap V(H^w_\varphi) = \emptyset$ if and only if $e_i \cap V(H^w_\varphi) = \emptyset$ and consequently $e_0 \cap V(H^w_\varphi) = \emptyset$ if and only if $e_i \cap V(H^w_\varphi) = \emptyset$ for all $i = 1, \ldots, k$. By setting $e_x := e_k$, the assertion follows. □



(a) Hypergraph H

(b) $H_{\varphi_1}/\mathcal{P}^R_{\varphi_1}$ (c) $H_{\varphi_2}/\mathcal{P}^R_{\varphi_2}$

(d) $H/\mathcal{P}^R = H_{\varphi_1}/\mathcal{P}^R_{\varphi_1} \square H_{\varphi_2}/\mathcal{P}^R_{\varphi_2}$

Fig. 7.3: The equivalence relation R on $E(H)$ with equivalence classes φ_1 (solid), φ_2 (dashed) has the relaxed grid property. We have $\mathcal{P}^R_{\varphi_1} = \{\{1,5,9\},\{2,6,10\},\{3,7,11\},\{4,8,12\}\}$, $\mathcal{P}^R_{\varphi_2} = \{\{1,\ldots,4,9,\ldots,12\},\{5,\ldots,8\}\}$, and $\mathcal{P}^R = \{\{1,9\},\{2,10\},\{3,11\},\{4,12\}\}$. The corresponding quotient graphs $H_{\varphi_i}/\mathcal{P}^R_{\varphi_i}$, $i = 1,2$ and the product graph $H/\mathcal{P}^R$ are shown on the right-hand side.

Proof of Theorem 7.20. Let $\varphi_1, \dots \varphi_n$ denote the equivalence classes of R. Analogously as in the graph case in Theorem 5.1, we show that

$$V_R(x) \mapsto (H^{v_1}_{\overline{\varphi}_1}, \dots, H^{v_n}_{\overline{\varphi}_n})$$

iff $x \in V(H^{v_i}_{\overline{\varphi}_i})$ for all $i = 1, \dots n$, defines a bijection $V(H/\mathcal{P}^R) \to V(\Box_{i=1}^n H_{\varphi_i}/\mathcal{P}^R_{\overline{\varphi}_i})$.

It remains to prove the isomorphism property, i.e., that $\{V_R(x_1), \dots, V_R(x_k)\}$ is an edge in $E(H/\mathcal{P}^R)$ if and only if $\{(H^{x_1}_{\overline{\varphi}_1}, \dots, H^{x_1}_{\overline{\varphi}_n}), \dots, (H^{x_k}_{\overline{\varphi}_1}, \dots, H^{x_k}_{\overline{\varphi}_n})\}$ is an edge in $\Box_{i=1}^n H_{\varphi_i}/\mathcal{P}_{\overline{\varphi}_i}$.

Let $\{V_R(x_1), \dots, V_R(x_k)\}$ be an edge in $E(H/\mathcal{P}^R)$. Thus, there exists an edge $e \in E(H)$ such that $e \cap V_R(x_j) \neq \emptyset$ for all $j = 1, \dots, k$ and $e \subseteq \bigcup_{j=1}^k V_R(x_j)$. Clearly, $e \in \varphi_m$ for some $m \in \{1, \dots, n\}$, and hence, $e \in \overline{\varphi}_l$ for all $l \neq m$, which implies $H^{x_1}_{\overline{\varphi}_l} = H^{x_j}_{\overline{\varphi}_l}$ for all $j = 1, \dots, k$ and all $l \neq m$. We have to show that $\{H^{x_1}_{\overline{\varphi}_m}, \dots, H^{x_k}_{\overline{\varphi}_m}\}$ is an edge in $H_{\overline{\varphi}_m}/\mathcal{P}^R_{\overline{\varphi}_m}$. Recall, that $V_R(x_j) = \bigcap_{i=1}^n V(H^{x_j}_{\overline{\varphi}_i})$ and consequently, $e \cap V_r(x_j) \neq \emptyset$ implies $e \cap V(H^{x_j}_{\overline{\varphi}_m}) \neq \emptyset$, as well as $e \subseteq \bigcup_{j=1}^k V_R(x_j)$ implies $e \subseteq \bigcup_{j=1}^k V(H^{x_j}_{\overline{\varphi}_i})$. Thus, by definition of quotient hypergraphs we have $\{H^{x_1}_{\overline{\varphi}_m}, \dots, H^{x_k}_{\overline{\varphi}_m}\} \in E(H_{\overline{\varphi}_m}/\mathcal{P}^R_{\overline{\varphi}_m})$ and hence $\{(H^{x_1}_{\overline{\varphi}_1}, \dots, H^{x_1}_{\overline{\varphi}_n}), \dots, (H^{x_k}_{\overline{\varphi}_1}, \dots, H^{x_k}_{\overline{\varphi}_n})\}$ is an edge in $\Box_{i=1}^n H_{\varphi_i}/\mathcal{P}_{\overline{\varphi}_i}$.

Now, let $\{(H^{x_1}_{\overline{\varphi}_1}, \dots, H^{x_1}_{\overline{\varphi}_n}), \dots, (H^{x_k}_{\overline{\varphi}_1}, \dots, H^{x_k}_{\overline{\varphi}_n})\}$ be an edge in $\Box_{i=1}^n H_{\varphi_i}/\mathcal{P}_{\overline{\varphi}_i}$. Then there exists some $m \in \{1, \dots, n\}$ such that $H^{x_1}_{\overline{\varphi}_l} = H^{x_j}_{\overline{\varphi}_l}$ for all $j = 1, \dots, k$ and all $l \neq m$ and $\{H^{x_1}_{\overline{\varphi}_m}, \dots, H^{x_k}_{\overline{\varphi}_m}\} \in E(H_{\overline{\varphi}_m}/\mathcal{P}^R_{\overline{\varphi}_m})$. That is, there exists $e \in \varphi_m$ such that $e \cap V(H^{x_j}_{\varphi_m}) \neq \emptyset$ for all $j = 1, \dots, k$ and $e \subseteq \bigcup_{j=1}^k V(H^{x_j}_{\varphi_m})$. Hence, there exists $x \in e \cap H^{x_1}_{\varphi_m}$. By Lemma 7.21, we can conclude that there exists an edge $e' \in \varphi_m$ such that $x_1 \in e'$ and $e' \cap V(H^{x_j}_{\overline{\varphi}_m}) \neq \emptyset$ for all $j = 1, \dots, k$ and $e' \cap V(H^{w}_{\overline{\varphi}_m}) = \emptyset$ if $w \notin V(H^{x_j}_{\overline{\varphi}_m})$ for some $j \in \{1, \dots, k\}$. Let $z_j \in e' \cap V(H^{x_j}_{\overline{\varphi}_m})$. Consequently, $z_j \in V(H^{x_1}_{\overline{\varphi}_l}) = V(H^{x_j}_{\overline{\varphi}_l})$ for all $l \neq m$, hence $z_j \in V_R(x_j)$, and therefore $e' \cap V_R(x_j) \neq \emptyset$ for all $j = 1, \dots, n$. Furthermore, since $e' \cap V(H^{w}_{\overline{\varphi}_m}) = \emptyset$ if $w \notin V(H^{x_j}_{\overline{\varphi}_m})$ for $j \in \{1, \dots, k\}$, we have $e' \subseteq \bigcup_{j=1}^k V(x_j)$ and consequently $\{V_R(x_1), \dots, V_R(x_k)\} \in E(H/\mathcal{P}^R)$, completing the proof. $\square$

The following result, which establishes the converse of Lemma 7.14, provides a surprisingly simple characterization of product relations consisting of exactly two classes.

Proposition 7.22. *Let R be an equivalence relation on $E(H)$ with the relaxed grid property consisting only of equivalence classes φ and $\overline{\varphi}$. Then $|V(H^x_\varphi) \cap V(H^y_{\overline{\varphi}})| = 1$ holds for all $x, y \in V(H)$ if and only if R is a product relation.*

The proof of Proposition 7.22 is essentially the same as that of Proposition 5.5 for the analogous result for simple graphs.

7.3 Hypergraph Bundles

In the graph case, the analogs of Theorem 7.20 and Proposition 7.22 are intimately related to graph bundles [55], which intuitively can be seen as generalizations of products in the sense they consist of isomorphic fibers held together by a collection of squares. The grid property thus can be expected to play an important role for hypergraph bundles. We will explore this topic in detail in this section.

Definition 7.23 ((Cartesian) Hypergraph Bundle)**.** *Let B and F be hypergraphs. A hypergraph H is a* (Cartesian) hypergraph bundle *with fiber F over the base graph B if there is a mapping $p : H \to B$ which satisfies the following conditions:*

(1) p is a weak homomorphism.

(2) For each vertex $v \in V(B)$ holds $p^{-1}(v) \cong F$, and for each edge $e \in E(B)$ holds $p^{-1}(e) \cong e\Box F$.

The triple (H, p, B) is then called bundle presentation *of H.*

Note, $p^{-1}(e) \cong e\Box F$ means that edges e' of H with $p(e') = e$ are precisely the e-layers w.r.t. $e\Box F$.

We will see next, how hypergraph bundles can be constructed, and will show later on, that these two definitions are equivalent.

Definition 7.24. *Let F and B be hypergraphs. Furthermore, let $\sigma : V(B) \times V(B) \to Aut(F)$ such that for all $u \in V(B)$ holds $\sigma(u, u) = id$ and for all $e \in E(B)$ and $u, v, w \in e$ holds $\sigma(v, w) \circ \sigma(u, v) = \sigma(u, w)$. For brevity, we will write σ_{uv} instead of $\sigma(u, v)$. We now define hypergraph H as follows:*

$$V(H) = V(B) \times V(H) \tag{7.2}$$

$$\begin{aligned} E(H) = &\{\{b\} \times e_F \mid b \in V(B), e_F \in E(F)\} \\ &\cup \{\{(b_1, f), (b_2, \sigma_{b_1b_2}(f)), \dots, (b_k, \sigma_{b_1b_k}(f))\} \\ &\mid \{b_1, \dots, b_k\} \in E(B), k \leq r(B), f \in V(F)\}. \end{aligned} \tag{7.3}$$

Proposition 7.25. *A hypergraph H is a hypergraph bundle with fiber F over base hypergraph B if and only if it can be constructed from B and F as in Definition 7.24.*

Proof. First we show, that any graph constructed as in Definition 7.24 satisfies the conditions of Definition 7.23. Therefore, let H be a hypergraph with $V(H) = V(B) \times V(F)$ and edges as in Equation (7.3). Let p denote the projection of $V(H)$ onto $V(B)$, i.e.: $p((b, f)) = f$ for all $b \in V(B)$, $f \in V(F)$. By construction, for the edges in H holds $p(\{b\} \times e_F) = b \in V(B)$ and $p(\{\{(b_1, f), (b_2, \sigma_{b_1,b_2}(f)), \dots, (b_k, \sigma_{b_1,b_k}(f))\}\}) = \{b_1, \dots, b_k\} \in E(B)$. Thus, p is a weak homomorphism. Moreover, it is easy to verify that for any $b \in V(B)$ the mapping $(b, f) \mapsto f$ for all $f \in V(F)$ defines an isomorphism $p^{-1}(b) = \langle\{(b, f) \mid f \in V(F)\}\rangle \cong F$.

Consider $p^{-1}(e)$ for $e = \{b_1, \dots, b_k\} \in E(B)$. it has vertex set $\{(b_i, \sigma_{b_1,b_i}(f)) \mid i = 1, \dots, k, f \in V(F)\} = \{(b_i, f) \mid i = 1, \dots, k, f \in V(F)\} = V(e) \times V(F)$. We define a mapping $\imath : e\Box F \to p^{-1}(e)$ by $\imath((b_i, f)) = (b_i, \sigma_{b_1,b_i}(f))$ for all $i = 1, \dots, k$ and all $f \in V(F)$. Obviously, $\imath$ is bijective and $\imath^{-1}$ is given by $\imath^{-1}((b_i, f)) = (b_i, \sigma_{b_i,b_1}(f))$. Thus, it remains to verify the isomorphism property. Therefore let e' be an edge in $e\Box F$. If $e' = e \times \{f\}$ for an $f \in V(F)$, we have $\imath(e') = \{(b_1, f), (b_2, \sigma_{b_1,b_2}(f)), \ldots, (b_k, \sigma_{b_1,b_k}(f))\} \in E(p^{-1}(e))$. If $e' = \{b_i\} \times e_F$ with $e_F = \{f_1, \ldots, f_l\} \in E(F)$, we have $\imath(e') = \{(b_i, \sigma_{b_1,b_i}(f_1)), \ldots, (b_i, \sigma_{b_1,b_i}(f_l))\}$ and since $\sigma_{b_1,b_i} \in Aut(F)$ it follows $\{\sigma_{b_1,b_i}(f_1), \ldots, \sigma_{b_1,b_i}(f_l)\} = e'_F \in E(F)$ hence, $\imath(e') = \{b_i\} \times e'_F \in E(p^{-1}(e))$. Conversely let e' be an edge in $p^{-1}(e)$. If it is of the form $e' = \{(b_1, f), (b_2, \sigma_{b_1,b_2}(f)), \ldots, (b_k, \sigma_{b_1,b_k}(f))\}$, we have $\imath^{-1}(e') = \{(b_1, f), (b_2, f), \ldots, (b_k, f)\} \in E(e\Box F)$. If $e' = \{b_i\} \times e_F = \{(b_i, f_1), \ldots, (b_i, f_l)\}$ for $e_F = \{b_1, \ldots, b_l\} \in E(F)$, it follows $\imath^{-1}(e') = \{(b_i, \sigma_{b_i,b_1}(f_1)), \ldots, (b_i, \sigma_{b_i,b_1}(f_l))\}$. Since $\sigma_{b_i,b_1} \in Aut(F)$, we have $\{\sigma_{b_i,b_1}(f_1), \ldots, \sigma_{b_i,b_1}(f_l)\} = e'_F \in E(F)$ an thus, $\imath^{-1}(e) = \{b_i\} \times e'_F \in E(e\Box F)$. Hence, H is a hypergraph bundle.

Now, we have to show that every hypergraph bundle can be constructed as in Definition 7.24. From Condition (2) in Definition 7.23, we can infer $|V(H)| = |V(B)||V(F)| = |V(B) \times V(F)|$. Thus, we can assign coordinates $h = (c_1(h), c_2(h))$ to each $h \in V(H)$ with $c_1(h) = p(h)$ and $c_2(h) \in V(F)$ for all $h \in V(H)$ such that for all $b \in V(B)$ the isomorphism $F \cong p^{-1}(b)$ is given by

$$f \mapsto (b, \sigma_b(f))$$

for some $\sigma_b \in Aut(F)$. Let $e = \{b_1, \ldots, b_k\} \in E(B)$ be arbitrary chosen and consider $p^{-1}(e) \cong e\Box F$. Recall, that the e-layers w.r.t. $e\Box F$ are just the edges $e' \in E(H)$ with $p(e') = e$. Hence, the F-layers are just given by $p^{-1}(v)$ for $v \in e$. Thus, we can choose a coordinatization of $V(p^{-1}(e))$ w.r.t. Cartesian product such that $p(h) = c_1(h)$ for all $h \in V(p^{-1}(e))$. Hence, the mapping

$$(b_i, f) \mapsto (b_i, \sigma_{b_i}(f))$$

defines an isomorphism $e\Box F \cong p^{-1}(e)$. Thus, for the edge set $E(p^{-1}(e))$ holds: $\{b_i\} \times \sigma_{b_i}(e_f) \in E(p^{-1}(e))$ for all $i = 1, \ldots, k$ and all $e_f \in E(F)$ and since $\sigma_{b_i} \in Aut(F)$, this is equivalent to $\{b_i\} \times \sigma_{b_i}(e_f) \in E(p^{-1}(e))$ for all $i = 1, \ldots, k$ and all $e_f \in E(F)$. Furthermore, $\{(b_1, \sigma_{b_1}(f)), \ldots, (b_k, \sigma_{b_k}(f))\} \in E(p^{-1}(e))$ for all $f \in V(F)$, which is equivalent to $\{(b_1, f), (b_2, (\sigma_{b_2} \circ \sigma_{b_1}^{-1})(f)), \ldots, (b_k, (\sigma_{b_k} \circ \sigma_{b_1}^{-1})(f))\} \in E(p^{-1}(e))$. With $\sigma_{b_i} \circ \sigma_{b_1}^{-1} =: \sigma_{b_1,b_i}$, and since this holds for all $e \in E(H)$, the assertion follows. □

Lemma 7.26. *Let R be an equivalence relation on the edge set of a connected hypergraph H that satisfies the grid property and has only two equivalence classes φ and $\overline{\varphi}$. Let p_φ denote the canonical mapping $p_\varphi : H \to H/\mathcal{P}_\varphi$. Then for any edge $e \in E(H/\mathcal{P}_\varphi)$ holds: The restriction of R to $E(p_\varphi^{-1}(e))$ satisfies the grid property.*

Proof. Let $e = \{H_\varphi^{x_1}, \ldots, H_\varphi^{x_r}\} \in E(H/\mathcal{P}_\varphi)$. Then $V(p_\varphi^{-1}(e)) = \bigcup_{i=1}^r V(H_\varphi^{x_i})$ and $E(p_\varphi^{-1}(e)) = \bigcup_{i=1}^r E(H_\varphi^{x_i}) \cup \{e' \in \overline{\varphi} \mid p_\varphi(e') = e\}$. We have to show, that any pair of adjacent edges $g, h \in E(p_\varphi^{-1}(e))$ with $g \in \varphi$, $h \in \overline{\varphi}$ span exactly one diagonal free grid in $p_\varphi^{-1}(e)$. That is, we have to show that the unique grid $\mathcal{G} = \{g, g_1, \ldots, g_{|h|-1}, h, h_1, \ldots, h_{|g|-1}\}$ spanned by g and h in H is contained in $p_\varphi^{-1}(e)$. By construction, we have $h \cap V(H_\varphi^x) \neq \emptyset$

if and only if $x \in V(H_\varphi^{x_i})$ for some $i \in \{1,\dots,r\}$. Furthermore, for all $j = 1,\dots,|h|-1$ holds $g_j \in \varphi$ implies $g_j \in E(H_\varphi^{x_i})$ for some $i \in \{1,\dots,r\}$. Thus, $g_j \in E(p_\varphi^{-1}(e))$ for all $j = 1,\dots,|h|-1$. Moreover, by definition of a grid we can conclude $h_j \cap V(H_\varphi^x) \neq \emptyset$ if and only if $h \cap V(H_\varphi^x) \neq \emptyset$, hence, $h_j \in E(p_\varphi^{-1}(e))$ for all $j = 1,\dots,|g|-1$, which completes the proof. □

The following theorem generalizes a well known result for graph bundles established in [46].

Theorem 7.27. *Let R be an equivalence relation on the edge set of a connected hypergraph H that satisfies the grid property. If φ is a 2-convex equivalence class of R, then $(H, p_\varphi, H_{\overline{\varphi}}/\mathcal{P}_\varphi^R)$ is a hypergraph bundle presentation.*

Proof. By construction, $p_\varphi : H \to H_{\overline{\varphi}}/\mathcal{P}_\varphi^R$ is a weak homomorphism. Therefore, it suffices to show $p_\varphi^{-1}(e) \cong e \Box H_\varphi^x$ holds for all $e \in E(H_{\overline{\varphi}}/\mathcal{P}_\varphi^R)$ such that edges mapped to e reflect the layers w.r.t. $e \Box H_\varphi^x$. From that we conclude $p^{-1}(v) \cong H_\varphi^x$ for all $v \in e$, hence, by connectivity for all $v \in V(H_{\overline{\varphi}}/\mathcal{P}_\varphi^R)$.

Let $e \in E(H_{\overline{\varphi}}/\mathcal{P}_\varphi^R)$. By Lemma 7.26 and Proposition 7.22, it suffices to show that $e' \cap e'' = \emptyset$ for all e', e'' with $p_\varphi(e') = p_\varphi(e'') = e$ and $|V(H_\varphi^x) \cap e'| = 1$ for all $x \in V(p^{-1}(e))$ and all e' with $p_\varphi(e') = e$ if φ is 2-convex. By construction, we have $e', e'' \in \overline{\varphi}$. Suppose, $e' \cap e'' \neq \emptyset$. Let $x \in e' \cap e''$. Since $e' \neq e''$, and H is simple, there exists $u, v \in V(p_\varphi^{-1}(e))$ such that $u \in e' \setminus e''$ and $v \in e'' \setminus e'$. By Lemma 7.21, we can choose $v \in V(H_\varphi^u)$ and since φ is 2-convex, there must be an edge $f \in \varphi$ such that $u, v \in f$. By the grid property, f and e' span a unique grid in H. But then e'' would be a diagonal of this grid, a contradiction. By Lemma 7.21, we can conclude, $V(H_\varphi^x) \cap e' \neq \emptyset$ holds for all $x \in V(p^{-1}(e))$ and all e' with $p_\varphi(e') = e$. If $|V(H_\varphi^x) \cap e'| > 1$ for some $x \in V(p^{-1}(e))$ and e' with $p_\varphi(e') = e$, H_φ^x would not be 1-convex and therefore not 2-convex. □

From Lemmas 6.1, 7.17, 7.19 and Remark 4.16, we can infer the following relation to the 2-section of a hypergraph bundle:

Corollary 7.28. *Let H be a hypergraph, R an equivalence relation on $E(H)$ that has the strong grid property, and let $\varphi \sqsubseteq R$ be 2-convex. Then $[H]_2$ is a hypergraph bundle over simple base graph $[H_{\overline{\varphi}}/P_\varphi^R]_2$.*

Note, if H is a hypergraph bundle over the simple base hypergraph B, this does not imply that $[H]_2$ is a graph bundle over simple base graph $[B]_2$ in general. As an example, see Figure 7.1. However, we can interpret the 2-section of a hypergraph as a multigraph, where an edge between two vertices is added in the 2-section for each edge of the hypergraph, that contains these two vertices. Thus, the multiplicity of an edge in the 2-section is just the cardinality of edges in the corresponding hypergraph that contains both endpoints of this edge. It follows then from Lemma 6.2 that $[H]_2$ is a (not necessarily simple) graph bundle over (not necessarily simple) base graph $[B]_2$ whenever H is a hypergraph bundle over base hypergraph B. The mapping $p' : [H]_2 \to [B]_2$ is then induced by $p : H \to B$, where (H, p, B) is the respective bundle presentation, such that $p'(v) = p(v)$ for all $v \in V(H) = V([H]_2)$. An edge e' with endpoints in $[H]_2$ $x, y \in V(H)$ is mapped to an edge f' in $[B]_2$ with endpoints $p(x), p(y) \in V(B)$ if e' results from the edge e in H, f' results from the edge f in B and $p(e) = f$.

Chapter 8

An Application of the 2-section: The Cartesian Skeleton and PFD Uniqueness Results of Strong Hypergraph Products

For a graph G, the key idea of finding its PFD with respect to the strong product is to find the PFD of a subgraph $\mathbb{S}(G)$ of G, the so-called *Cartesian skeleton*, with respect to the Cartesian product and construct the prime factors of G using the information of the PFD of $\mathbb{S}(G)$. This concept was first introduced for graphs by Feigenbaum and Schäffer in [16] and later on improved by Hammack and Imrich, see [27].

As it is shown in [37], it is possible to find several non-equivalent generalizations of the strong graph product to hypergraphs. In this chapter, we are concerned with two generalizations of the strong graph product, namely, the so-called normal product [60] and the strong (hypergraph) product [37]. We show in Section 8.2 that every connected simple thin hypergraph has a unique PFD with respect to these two products. For this purpose, we introduce the notion of the Cartesian skeleton of hypergraphs in Section 8.1 as a generalization of the Cartesian skeleton of graphs and show that it is uniquely defined for thin hypergraphs. As it turns out, the 2-section of a hypergraph seems to be a very useful tool for computing its Cartesian skeleton.

8.1 The Cartesian Skeleton

Following the approach of Hammack and Imrich, one removes edges in G that fulfill so-called dispensability conditions, resulting in a subgraph $\mathbb{S}(G)$ that is the desired Cartesian skeleton. The underlying concept of *dispensability* as defined for graphs in [27] can be generalized in a natural way for hypergraphs.

Definition 8.1 (Dispensability)**.** *An edge $e \in E(H)$ is* dispensable *in H if there exists a vertex $z \in V(H)$ and distinct vertices $x, y \in e$ for which both of the following statements hold:*

1. $N[x] \cap N[y] \subset N[x] \cap N[z]$ *or* $N[x] \subset N[z] \subset N[y]$
2. $N[x] \cap N[y] \subset N[y] \cap N[z]$ *or* $N[y] \subset N[z] \subset N[x]$.

Note, the latter definition coincides with the one given in [27], if H is a simple graph. Now, we are able to define the Cartesian skeleton for hypergraphs.

Definition 8.2 (Cartesian Skeleton)**.** *Let $D(H) \subseteq E(H)$ be the set of dispensable edges in a given hypergraph H. The* Cartesian skeleton *of a hypergraph H is the partial hypergraph $\mathbb{S}(H) \subseteq H$ where all dispensable edges $D(H)$ are removed from H, that is $V(\mathbb{S}(H)) = V(H)$ and $E(\mathbb{S}(H)) = E(H) \setminus D(H)$.*

In the next proposition, we shortly summarize the results established by Hammack and Imrich [27] concerning the Cartesian skeleton of graphs and show in the sequel, that these results can easily be transferred to hypergraphs by usage of its corresponding 2-sections.

Proposition 8.3 ([27])**.** *Let $G = G_1 \boxtimes G_2$ be a strong product graph.*

1. *If G is thin then every non-dispensable edge $e \in E(G)$ is Cartesian w.r.t. any factorization $G'_1 \boxtimes G'_2$ of G.*
2. *If G is connected, then $\mathbb{S}(G)$ is connected.*
3. *If G_1 and G_2 are thin graphs then $\mathbb{S}(G_1 \boxtimes G_2) = \mathbb{S}(G_1) \Box \mathbb{S}(G_2)$.*
4. *Any isomorphism $\varphi : G \to H$, as a map $V(G) \to V(H)$, is also an isomorphism $\varphi : \mathbb{S}(G) \to \mathbb{S}(H)$.*

Since neighborhoods of vertices in a hypergraph and its 2-section are identical by Lemma 6.7 and dispensability is defined only in terms of neighborhoods, we easily obtain the following lemma and corollary.

Lemma 8.4. *Let H be a hypergraph. The edge $e \in E(H)$ is dispensable in H if and only if there is an edge $e' \in E([H]_2)$ with $e' \subseteq e$ and e' is dispensable in $[H]_2$.*

Corollary 8.5. *For all hypergraphs H holds: $[\mathbb{S}(H)]_2 = \mathbb{S}([H]_2)$.*

From the Distance Formula and Proposition 8.3 we obtain immediately:

Corollary 8.6. *For all hypergraphs H holds: If H is connected then $\mathbb{S}(H)$ is connected.*

Lemma 8.7. *Let H be a hypergraph and $H_1 \boxtimes H_2$ be an arbitrary factorization of H. Then it holds that the edge e is Cartesian in H w.r.t. $H_1 \boxtimes H_2$ if and only if e' is Cartesian in $[H_1]_2 \boxtimes [H_2]_2 = [H]_2$ for all $e' \subseteq e$ with $e' \in E([H]_2)$.*

Proof. Let $e \in E(H)$ be Cartesian w.r.t. to its factorization $H_1 \boxtimes H_2$. Then, there is an $i \in \{1,2\}$ with $|p_i(e)| = 1$. Moreover, for all $e' \subseteq e$ it holds, $0 < |p_i(e')| \leq |p_i(e)| = 1$ and hence, $|p_i(e')| = 1$. Therefore, each edge $e' \in E([H]_2)$ with $e' \subseteq e$ is Cartesian in $[H]_2 = [H_1]_2 \boxtimes [H_2]_2$.

By contraposition, suppose $e \in E(H)$ is non-Cartesian w.r.t. $H_1 \boxtimes H_2$. Hence, by definition of the products $\breve{\boxtimes}$ and $\widehat{\boxtimes}$ we have $|p_i(e)| > 1$, $i = 1, 2$. Therefore, there are vertices $x, y \in e$ with $p_1(x) \neq p_1(y)$. If $p_2(x) \neq p_2(y)$ it follows that $e' = \{x, y\}$ is non-Cartesian in $[H]_2 = [H_1]_2 \boxtimes [H_2]_2$. If $p_2(x) = p_2(y)$ then there is a vertex $z \in e$ with $p_2(x) \neq p_2(z)$. If $p_1(z) \neq p_1(x)$ then the edge $e' = \{x, z\}$ is non-Cartesian in $[H]_2 = [H_1]_2 \boxtimes [H_2]_2$ and if $p_1(z) = p_1(x) \neq p_1(y)$ then the edge $e' = \{y, z\}$ is non-Cartesian in $[H]_2 = [H_1]_2 \boxtimes [H_2]_2$. □

Lemma 8.8. *Let H be a thin hypergraph. If $e \in E(H)$ is non-dispensable in H then the edge e is Cartesian w.r.t. any factorization $H_1 \boxtimes H_2$ of H.*

Proof. Let $e \in E(H)$ be non-dispensable in H. Lemma 8.4 implies that for all $e' \in E([H]_2)$ with $e' \subseteq e$ holds e' is non-dispensable in $[H]_2$. Furthermore, by Lemma 6.7 it holds that $[H]_2$ is thin. Thus, Proposition 8.3 implies that e' is Cartesian in $[H]_2$ for all $e' \subseteq e$, which is by Lemma 8.7 if and only if e is Cartesian in H. □

Proposition 8.9. *If H_1 and H_2 are thin hypergraphs, then $\mathbb{S}(H_1 \boxtimes H_2) = \mathbb{S}(H_1) \Box \mathbb{S}(H_2)$.*

Proof. Let $H = H_1 \boxtimes H_2$. Lemma 8.8 implies that every non-Cartesian edge is dispensable. Hence we need to show, that a Cartesian edge $e \in E(H)$ is dispensable if and only if $p_i(e)$ is dispensable whenever $p_i(e) \in E(H_i)$, $i = 1, 2$. Note, exactly for one $i \in \{1,2\}$ holds $p_i(e) \in E(H_i)$ and $p_j(e) \in V(H_j)$, $j \neq i$. W.l.o.g. assume $p_1(e) = e_1 \in E(H_1)$ and $p_2(e) = v_2 \in V(H_2)$.

Assume that the edge e is dispensable in H. Then by Lemma 8.4 there exists a dispensable edge $e' \in E([H]_2)$ with $e' \subseteq e$. Lemma 6.7 implies that $[H]_2$ is thin and by Proposition 8.3 it holds that $\mathbb{S}([H]_2) = \mathbb{S}([H_1]_2) \Box \mathbb{S}([H_2]_2)$ and hence, we infer $p_1(e')$ must be dispensable in $[H_1]_2$. Since $p_1(e') \subseteq e_1$ and by Lemma 8.4, we conclude that e_1 is dispensable in H_1.

Now suppose e_1 is dispensable in H_1. Again, by Lemma 8.4, there exists a dispensable edge $e_1' \in E([H_1]_2)$ such that $e_1' \subseteq e_1$. By Lemma 6.7 it holds that $[H]_2$ is thin and Proposition 8.3 implies $\mathbb{S}([H]_2) = \mathbb{S}([H_1]_2) \Box \mathbb{S}([H_2]_2)$. Therefore, $e' = e_1' \times \{v_2\}$ is dispensable in $[H]_2$. By Lemma 8.4 and since $e' \subseteq e$, we have e is dispensable in H. □

As in [27] the Cartesian skeleton $\mathbb{S}(H)$ is defined entirely in terms of the adjacency structure of H, and thus, we obtain the following immediate consequence of the definition.

Proposition 8.10. *Any isomorphism $\varphi : H \to G$, as a map $V(H) \to V(G)$, is also an isomorphism $\varphi : \mathbb{S}[H] \to \mathbb{S}[G]$.*

8.2 Prime Factorization Theorem

In the following, let $\boxtimes \in \{\breve{\boxtimes}, \widehat{\boxtimes}\}$. Let $A \boxtimes B$ and $C \boxtimes D$ be two non-trivial decompositions of a simple connected thin hypergraph H. We will show that then H has a finer factorization of the form $AC \boxtimes AD \boxtimes BC \boxtimes BD$ and $A = AC \boxtimes AD$, $B = BC \boxtimes BD$, C=$AC \boxtimes BC$ and $D = AD \boxtimes BD$, see Prop. 8.18. Similar as for graphs [43, page 171-174], this can be used to show that every simple thin connected hypergraph has a unique prime factorization with respect to the normal and strong (hypergraph) product. We don't want to conceal the fact, that in the sequel of this section, we make frequent use of the same arguments as for graph products in [43] and [28].

By Proposition 8.9, it holds $\mathbb{S}(H) = \mathbb{S}(A)\Box\mathbb{S}(B) = \mathbb{S}(C)\Box\mathbb{S}(D)$. Let $\mathbb{S}(H) = \Box_{i\in I} H_i$ be the unique PFD of the Cartesian skeleton of H. Hence, the factors $\mathbb{S}(A)$, $\mathbb{S}(B)$, $\mathbb{S}(C)$ and $\mathbb{S}(D)$ are all products of or isomorphic to the Cartesian *prime* factors of $\mathbb{S}(H)$. Let I_A be the subset of the index set I with $V(A) = V(\Box_{i\in I_A} H_i)$. Analogously, the index sets I_B, I_C and I_D are defined.

In the following, we define the hypergraphs AC, AD, BC and BD and as it will turn out it holds $H \cong AC \boxtimes AD \boxtimes BC \boxtimes BD$. Therefore, it will be convenient to use only four coordinates $x = (x_{AC}, x_{AD}, x_{BC}, x_{BD})$ for every vertex $x \in V(H)$. With this notation, the projections $p_{AC} : V(H) \to V(AC)$, $p_{AD} : V(H) \to V(AD)$, $p_{BC} : V(H) \to V(BC)$, $p_{BD} : V(H) \to V(BD)$ are well-defined.

Moreover, the vertex set of AC is defined as $V(AC) = V(\Box_{i\in I_A\cap I_C} H_i)$. Analogously, the vertex sets of AD, BC and BD are defined. It will be shown that $A = AC \boxtimes AD$, $B = BC \boxtimes BD$, C=$AC \boxtimes BC$ and $D = AD \boxtimes BD$. Of course it is possible that not all of the intersections $I_A \cap I_C, I_A \cap I_D, I_B \cap I_C$ and $I_B \cap I_D$ are nonempty. Suppose that $I_B \cap I_D = \emptyset$ then $I_A \cap I_D \neq \emptyset$, since otherwise $I_D = \emptyset$. If in addition $I_A \cap I_C$ were empty, then $I_A = I_D$ and thus $I_B = I_C$, but then there would be nothing to prove. Thus, we can assume that all but possibly $I_B \cap I_D$ are nonempty and at least three of the four coordinates are nontrivial, that is to say, there are at least two vertices that differ in the first, second and third coordinates, but it is possible that all vertices have the same fourth coordinate.

With the definition of the projections p_A, p_B, p_C and p_D together with the preceding construction of the coordinates $(x_{AC}, x_{AD}, x_{BC}, x_{BD})$ for vertices $x \in V(H)$, we thus have

$$x_A = p_A(x) = p_A(x_{AC}, x_{AD}, x_{BC}, x_{BD}) = (x_{AC}, x_{AD}, -, -) =: (x_{AC}, x_{AD}) \in V(A),$$

$$x_B = p_B(x) = p_B(x_{AC}, x_{AD}, x_{BC}, x_{BD}) = (-, -, x_{BC}, x_{BD}) =: (x_{BC}, x_{BD}) \in V(B),$$

$$x_C = p_C(x) = p_C(x_{AC}, x_{AD}, x_{BC}, x_{BD}) = (x_{AC}, -, x_{BC}, -) =: (x_{AC}, x_{BC}) \in V(C),$$

$$x_D = p_D(x) = p_D(x_{AC}, x_{AD}, x_{BC}, x_{BD}) = (-, x_{AD}, -, x_{BD}) =: (x_{AD}, x_{BD}) \in V(D).$$

In this way, vertices of A, B, C and D are coordinatized. Thus, the projections $p'_{AC} : V(A) \to V(AC)$ and $p''_{AC} : V(C) \to V(AC)$ are well-defined. Since for all $x \in V(H)$ holds that

$$p_{AC}(x) = p'_{AC}(p_A(x)) = p'_{AC}(x_A) = p''_{AC}(p_C(x)) = p''_{AC}(x_C) = x_{AC},$$

we will identify p_{AC} with p'_{AC}, resp., p''_{AC}, henceforth and simply write p_{AC}. Analogously, we identify the respective projections onto AD, BC and BD with p_{AD}, p_{BC}, p_{BD}.

We are now in the position to give the complete definition of the hypergraphs AC, AD, BC and BD. The vertex set of AC is

$$V(AC) = V(\Box_{i\in I_A\cap I_C} H_i) = \bigtimes_{i\in I_A\cap I_C} V(H_i) \tag{8.1}$$

The edge set of AC is

$$\begin{aligned} E(AC) = \{e_{AC} \subseteq V(AC) \mid \exists e_H \in E(H) \text{ with } p_{AC}(e_H) = e_{AC} \\ \text{s.t. } \nexists e'_H \in E(H) : p_{AC}(e_H) \subset p_{AC}(e'_H)\} \end{aligned} \tag{8.2}$$

Analogously, the hypergraphs AD, BC and BD are defined.

Equation (8.2), that characterizes the edge sets for the (putative) finer factors AC, AD, BC and BD w.r.t. $\boxtimes$, forces edges to be maximal with respect to inclusion. We need this definition, in particular for defining the factors of the normal product, since projections of edges into the factors might be proper subsets of edges different from a single vertex.

Remark 8.11. *Note, that vertices x are well defined by their entries x_{AC}, x_{AD}, x_{BC} and x_{BD} of their coordinates, independently from the ordering of x_{AC}, x_{AD}, x_{BC} and x_{BD}, since the coordinates will be clearly marked. Therefore, we henceforth distinguish vertices just by the entries of their coordinates rather than by the ordering.*

Lemma 8.12. *Let $H \cong A \boxtimes B \cong C \boxtimes D$ be a thin hypergraph and AC be as defined in Equations* (8.1) *and* (8.2). *Then it holds:*

1. *$e_{AC} \subseteq p_{AC}(e_A)$ implies $e_{AC} = p_{AC}(e_A)$ and $e_{AC} \subseteq p_{AC}(e_C)$ implies $e_{AC} = p_{AC}(e_C)$ for all edges $e_{AC} \in E(AC)$, $e_A \in E(A)$ and $e_C \in E(C)$.*

2. *If $p_{AC}(e_H) \in E(AC)$ then $p_A(e_H) \in E(A)$ and $p_C(e_H) \in E(C)$ for every edge $e_H \in E(H)$.*

Analogous results hold for the hypergraphs AD, BC and BD with respective edges.

Proof. For the proof of the first statement, let $e_{AC} \in E(AC)$ and suppose for contradiction, that there is an edge $e_A \in E(A)$ with $e_{AC} \subset p_{AC}(e_A)$. Thus, there is an edge $e_H \in E(H)$ with $e_H = e_A \times \{x_B\}, x_B \in V(B)$ and therefore, $e_{AC} \subset p_{AC}(e_H)$, which contradicts the definition of AC. Analogously, there is no edge $e_C \in E(C)$. such that $e_{AC} \subset p_{AC}(e_C)$.

For the proof of the second statement, let $e_H \in E(H)$ be an arbitrary edge and suppose that $p_{AC}(e_H) \in E(AC)$. Note, if $|p_{AC}(e_H)| > 1$ then there are at least two distinct vertices $x, x' \in e_H \in E(H)$ with $p_{AC}(x) = x_{AC} \neq p_{AC}(x') = x'_{AC}$. Hence, $p_A(x) \neq p_A(x')$ and $p_C(x) \neq p_C(x')$. Therefore, $|p_{AC}(e_H)| > 1$ implies that $|p_A(e_H)| > 1$ and $|p_C(e_H)| > 1$ for each edge $e_H \in E(H)$. Thus, whenever $p_{AC}(e_H) \in E(AC)$ then the projections $p_A(e_H)$ and $p_C(e_H)$ cannot be a single vertex.

If $\boxtimes = \widehat{\boxtimes}$ then the condition $p_A(e_H) \in E(A)$ and $p_C(e_H) \in E(C)$ is trivially fulfilled by the definition of $\widehat{\boxtimes}$, since $p_{AC}(e_H) \in E(AC)$ and thus, $|p_{AC}(e_H)| > 1$.

Now, consider the product $\breve{\boxtimes}$. Note, since $e_H \in E(e_A \breve{\boxtimes} e_B)$ for some $e_A \in E(A)$, $e_B \in E(B)$ we can conclude by definition of the normal product that $p_A(e_H) \subseteq e_A$ and thus, $p_{AC}(e_H) = p_{AC}(p_A(e_H)) \subseteq p_{AC}(e_A)$. By assumption, we have $p_{AC}(e_H) \in E(AC)$ and therefore, Item (1) of this lemma implies that $p_{AC}(e_H) = p_{AC}(e_A)$. Moreover, it holds that $|e_H| \geq |p_{AC}(e_H)|$ and by Remark 6.5 we have $|e_A| \geq |e_H| \geq |p_{AC}(e_H)|$. Since $H \cong A \breve{\boxtimes} B$ there is an edge $e'_H = e_A \times \{x_B\} \in E(H)$ which implies that $p_C(e'_H) = p_{AC}(e_A) \times \{x_{BC}\}$. Thus, $|p_C(e'_H)| = |p_{AC}(e_A)| \leq |e_A| = |e'_H|$, since e'_H is Cartesian w.r.t. $A \breve{\boxtimes} B$. Since $H \cong C \breve{\boxtimes} D$, and by the definition of the normal product, it holds $|p_C(e'_H)| = |e'_H|$, and therefore, $|e_A| = |p_{AC}(e_A)| = |p_{AC}(e_H)|$. Since $|e_A| \geq |e_H| \geq |p_{AC}(e_H)|$, it holds $|e_H| = |e_A|$. Thus, we can conclude by Remark 6.5 that $p_A(e_H) \in E(A)$. By similar arguments, one can show that $p_C(e_H) \in E(C)$. □

Lemma 8.13. *Let $H \cong A \boxtimes B \cong C \boxtimes D$ be a thin hypergraph and AC and BC be as defined in Equations* (8.1) *and* (8.2)*. Then for all $e_{AC} \in E(AC)$ and all $x_{BC} \in V(BC)$ there is an edge $e_C = e_{AC} \times \{x_{BC}\} \in E(C)$. Analogous results hold for the hypergraphs AD, BC and BD with respective edges.*

Proof. Let $e_{AC} \in E(AC)$ be an arbitrary edge. By definition of AC, there is an edge $e_H \in E(H)$ with $p_{AC}(e_H) = e_{AC}$. Note, by the same arguments as in the proof of Lemma 8.12 it holds that $|p_{AC}(e_H)| > 1$ implies $|p_A(e_H)| > 1$ and $|p_C(e_H)| > 1$ for each $e_H \in E(H)$.

Since $e_H \in E(A \boxtimes B)$, there is an edge $e_A \in E(A)$ s.t. $p_A(e_H) \subseteq e_A$. Therefore, $e_{AC} = p_{AC}(e_H) = p_{AC}(p_A(e_H)) \subseteq p_{AC}(e_A)$ which implies together with Lemma 8.12 (1), that $p_{AC}(e_A) = e_{AC}$. By Lemma 8.12 (2), we have $p_A(e_H) = e_A$. Therefore, there is an edge of the form $e_A \times \{x_B\} \in E(H)$. W.l.o.g. let us assume that e_H is chosen s.t. $e_H = e_A \times \{x_B\}$. Since we also have $e_H \in E(C \boxtimes D)$ there is an edge $e_C \in E(C)$ s.t. $p_C(e_H) \subseteq e_C$. Analogously, we can conclude by Lemma 8.12 $p_C(e_H) = e_C$. Hence, $e_C = p_{AC}(e_A) \times \{x_{BC}\} = e_{AC} \times \{x_{BC}\} \in E(C)$. □

Lemma 8.14. *Let $H \cong A \boxtimes B \cong C \boxtimes D$ be a thin hypergraph and AC and BC be as defined in Equation* (8.1) *and* (8.2)*. Then it holds that $p_{AC}(e_C) \in E(AC)$ for all edges $e_C \in E(C)$ with $e_C = p_{AC}(e_C) \times \{x_{BC}\}$, $x_{BC} \in V(BC)$. Analogous results hold for the hypergraphs AD, BC and BD with respective edges.*

Proof. Let $e_C = p_{AC}(e_C) \times \{x_{BC}\} \in E(C)$. Since $H \cong C \boxtimes D$, there is an edge $e_H = e_C \times \{x_D\} \in E(H)$. It holds $p_{AC}(e_C) = p_{AC}(p_C(e_H)) = p_{AC}(e_H) \subseteq e_{AC} \in E(AC)$. Suppose for contradiction, that $p_{AC}(e_C) \subset e_{AC} \in E(AC)$. Then there is by definition of AC another edge $e'_H \in E(H)$ with $p_{AC}(e'_H) = e_{AC}$. Since $H \cong A \boxtimes B$, there is an edge $e_A \in E(A)$ with $p_A(e'_H) \subseteq e_A$. Hence, we have $p_{AC}(e_H) = p_{AC}(e_C) \subset p_{AC}(e'_H) = p_{AC}(p_A(e'_H)) \subseteq p_{AC}(e_A)$, shortly, $p_{AC}(e_H) \subset p_{AC}(e_A)$. By definition of the normal and the strong product, there is an edge $e''_H = e_A \times \{x_B\} \in E(H)$. Since we assumed to have $e_C = p_{AC}(e_C) \times \{x_{BC}\}$ it holds $e_C \subset e_{AC} \times \{x_{BC}\} = p_C(e''_H) \subseteq e'_C$ for some $e'_C \in E(C)$ contradicting that C is simple. Thus, $p_{AC}(e_C) = e_{AC} \in E(AC)$. $\square$

Corollary 8.15. *Let $H \cong A \boxtimes B \cong C \boxtimes D$ be a thin hypergraph and AC,AD,BC and BD be as defined in Equations* (8.1) *and* (8.2)*. Then it holds that $e_{AC} \in E(AC)$ if and only if there is an edge $e_H \in E(H)$ with $e_H = e_{AC} \times \{x_{AD}\} \times \{x_{BC}\} \times \{x_{BD}\}$, $x_{AD} \in V(AD)$, $x_{BC} \in V(BC)$,$x_{BD} \in V(BD)$. Analogous results hold for respective edges of the hypergraphs AD, BC and BD.*

Proof. If $e_{AC} \in E(AC)$ then by Lemma 8.13 there is an edge $e_C = e_{AC} \times \{x_{BC}\} \in E(C)$. Since $H \cong C \boxtimes D$ and by choice of the coordinates, there is an edge $e_H = e_C \times \{x_D\} \in E(H)$ with $x_D = (x_{AD}, x_{BD})$. Hence, e_H can be written as $e_{AC} \times \{x_{AD}\} \times \{x_{BC}\} \times \{x_{BD}\}$.

If $e_H = e_{AC} \times \{x_{AD}\} \times \{x_{BC}\} \times \{x_{BD}\}$ it follows that $|p_B(e_H)| = 1$ and $|p_D(e_H)| = 1$ and thus, this edge e_H is Cartesian in $A \boxtimes B$ and $C \boxtimes D$. Therefore, $p_A(e_H) \in E(A)$ and $p_C(e_H) \in E(C)$. Now, suppose for contradiction that $e_{AC} \notin E(AC)$. By definition of AC, there is an edge e'_H with $p_{AC}(e'_H) \in E(AC)$ such that $e_{AC} = p_{AC}(e_H) \subset p_{AC}(e'_H)$. By Lemma 8.13 there is an edge $e_C = p_{AC}(e'_H) \times \{x_{BC}\}$ and hence, an edge $e''_H = p_{AC}(e'_H) \times \{x_{AD}\} \times \{x_{BC}\} \times \{x_{BD}\}$, which implies that $e_H \subset e''_H$, contradicting that H is simple. $\square$

Lemma 8.16. *Let $H \cong A \boxtimes B \cong C \boxtimes D$ be a thin hypergraph and AC, AD, BC and BD be as defined in Equations* (8.1) *and* (8.2)*. Then for all $e_{AC} \in E(AC)$, $e_{AD} \in E(AD)$ and $x_B \in V(B)$ it holds that $E(e_{AC} \boxtimes e_{AD}) \times \{x_B\} \subseteq E(H)$. Analogous results hold with respective edges in the hypergraphs BC and BD and vertices $x_A \in V(A), x_C \in V(C)$ and $x_D \in V(D)$.*

Proof. Let $x_B = (x_{BC}, x_{BD}) \in V(B)$ with $x_{BC} \in V(BC)$, $x_{BD} \in V(BD)$, $e_{AC} \in E(AC)$ and $e_{AD} \in E(AD)$. By Lemma 8.13 there is an edge $e_C = e_{AC} \times \{x_{BC}\} \in E(C)$ and analogously, there is also an edge $e_D = e_{AD} \times \{x_{BD}\} \in E(D)$. Hence, it holds: $E(e_{AC} \boxtimes e_{AD}) \times \{x_B\} = E(e_{AC} \boxtimes e_{AD} \boxtimes (\{x_{BC}\}, \emptyset) \boxtimes (\{x_{BD}\}, \emptyset)) = E((e_{AC} \times \{x_{BC}\}) \boxtimes (e_{AC} \times \{x_{BD}\})) = E(e_C \boxtimes e_D) \subseteq E(H)$. $\square$

Lemma 8.17. *Let $H \cong A \boxtimes B \cong C \boxtimes D$ be a thin hypergraph and AC and AD be as defined in Equations* (8.1) *and* (8.2)*. Then for all edges $e_A \in E(A)$ there is an edge $e_{AC} \in E(AC)$ and $e_{AD} \in E(AD)$ such that $e_A \in E(e_{AC} \boxtimes e_{AD})$. Analogous results hold for the hypergraphs B, C, D with respective edges from AC, AD, BC and BD, whenever $I_B \cap I_D \neq \emptyset$.*

Proof. Let $e_A \in E(A)$ and $x_B = (x_{BC}, x_{BD}) \in V(B)$. Since $H \cong A \boxtimes B$, there is a Cartesian edge $e_H = e_A \times \{x_B\} \in E(H)$. Furthermore, since $H \cong C \boxtimes D$ and by definition of the normal and the strong product, we can conclude that $p_C(e_H) \in V(C)$ or there is an edge $e_C \in E(C)$ with $p_C(e_H) \subseteq e_C$, as well as, $p_D(e_H) \in V(D)$ or there is an edge $e_D \in E(D)$ with $p_D(e_H) \subseteq e_D$.

Assume first $x_D = p_D(e_H) \in V(D)$. Then $p_C(e_H) = e_C \in E(C)$, that is, $e_H = e_C \times \{x_D\}$. Note, coordinates of vertices $x_C \in e_C$ are given by (x_{AC}, x_{BC}). Since $e_H = e_A \times \{x_B\} \in E(H)$ it holds that $p_{BC}(e_C) = p_{BC}(p_C(e_H)) = p_{BC}(e_H) = x_{BC}$. Therefore, e_H can be written as $p_{AC}(e_C) \times \{x_{BC}\} \times \{x_D\}$. Moreover, $p_{AC}(e_C) = p_{AC}(e_H) = p_{AC}(e_A)$ and hence, $p_C(e_H) = e_C = p_{AC}(e_A) \times \{x_{BC}\} \in E(C)$. Now, Lemma 8.14 implies that $p_{AC}(e_A) = e_{AC} \in E(AC)$. Moreover, it holds $p_{AD}(e_A) = p_{AD}(e_H) = p_{AD}(p_D(e_H)) = p_{AD}(x_D) = x_{AD} \in V(AD)$ and therefore, $e_A = e_{AC} \times \{x_{AD}\}$ and thus, $e_A \in E(e_{AC} \boxtimes e_{AD})$ for all e_{AD} with $\{x_{AD}\} \in e_{AD}$. Analogously, we infer that $e_A = \{x_{AC}\} \times e_{AD}$, $x_{AC} \in V(AC)$ and therefore, $e_A \in E(e_{AC} \boxtimes e_{AD})$ for all e_{AC} with $x_{AC} \in e_{AC}$ if $p_C(e_H) \in V(C)$.

Now, we treat the case $p_C(e_H) \subseteq e_C \in E(C)$ and $p_D(e_H) \subseteq e_D \in E(D)$ and consider the different products $\breve{\boxtimes}$ and $\widehat{\boxtimes}$ separately.

In case $\widehat{\boxtimes}$ we have, $p_C(e_H) = e_C = p_{AC}(e_H) \times \{x_{BC}\} \in E(C)$ and $p_D(e_H) = e_D = p_{AD}(e_H) \times \{x_{BD}\} \in E(D)$ and by the same arguments as before, $p_{AC}(e_H) = p_{AC}(e_A) = e_{AC} \in E(AC)$ and $p_{AD}(e_H) = p_{AD}(e_A) = e_{AD} \in E(AD)$. Since $e_H = e_A \times \{x_B\} \in E(e_C \,\widehat{\boxtimes}\, e_D)$ and $E(e_C \,\widehat{\boxtimes}\, e_D) = E(e_{AC} \,\widehat{\boxtimes}\, e_{AD}) \times \{x_B\}$ we can conclude that $e_A \in E(e_{AC} \,\widehat{\boxtimes}\, e_{AD})$.

In case $\breve{\boxtimes}$ we have, $p_{AC}(e_A) = p_{AC}(e_H) = p_{AC}(p_C(e_H)) \subseteq p_{AC}(e_c) = p_{AC}(e_C \times \{x_D\})$ with $e_C \times \{x_D\} \in E(H)$ and therefore $p_{AC}(e_A) \subseteq p_{AC}(e_C \times \{x_D\}) \subseteq e_{AC} \in E(AC)$. Analogously it holds $p_{AD}(e_A) \subseteq e_{AD} \in E(AD)$. Note, by definition of $\breve{\boxtimes}$ it holds $p_C(e_H) = e_C$ or $p_D(e_H) = e_D$. Lemma 8.14 implies that if $p_C(e_H) = e_C$ then $p_{AC}(e_A) = e_{AC}$ and if $p_D(e_H) = e_D$ then $p_{AD}(e_A) = e_{AD}$. Furthermore, it holds by definition of the normal product $|p_C(e_H)| = |p_D(e_H)|$. If $p_C(e_H) = e_C$ then, by the choice of e_H, we have $|e_{AC}| = |e_C| = |p_C(e_H)| = |p_D(e_H)| = |p_{AD}(e_A)| \leq |e_{AD}|$. If $p_D(e_H) = e_D$ we have $|e_{AD}| = |e_D| = |p_D(e_H)| = |p_C(e_H)| = |p_{AC}(e_A)| \leq |e_{AC}|$. Therefore, we can conclude that $|e_A| = |e_H| = \min\{|e_C|, |e_D|\} = \min\{|e_{AC}|, |e_{AD}|\}$ and thus, $e_A \in E(e_{AC} \,\breve{\boxtimes}\, e_{AD})$. $\square$

Proposition 8.18. *Let $H \cong A \boxtimes B \cong C \boxtimes D$ be a thin hypergraph. Then there exists a decomposition*

$$H \cong AC \boxtimes AD \boxtimes BC \boxtimes BD$$

of H such that $A = AC \boxtimes AD$, $B = BC \boxtimes BD$, $C = AC \boxtimes BC$ and $D = AD \boxtimes BD$.

Proof. First we show that there is a decomposition $AC \boxtimes AD$ of A. Let AC and AD be defined as in Equation (8.1) and (8.2). Thus, by construction of AC and AD we have $V(A) = V(AC) \times V(AD)$. Therefore, we need to show that $E(A) = E(AC \boxtimes AD)$.

By Lemma 8.17 and since $E(e_{AC} \boxtimes e_{AD}) \subseteq E(AC \boxtimes AD)$ for all $e_{AC} \in E(AC)$ and $e_{AD} \in E(AD)$ we have $E(A) \subseteq E(AC \boxtimes AD)$.

Let $e \in E(AC \boxtimes AD)$. Hence, there is an edge $e_{AC} \in E(AC)$ and $e_{AD} \in E(AD)$ with $e \in E(e_{AC} \boxtimes e_{AD})$. By Lemma 8.16 we can conclude that there is a vertex $x_B \in V(B)$ such that $e \times \{x_B\} \in E(e_{AC} \boxtimes e_{AD}) \times \{x_B\} \subseteq E(H)$. Since $e = p_A(e \times \{x_B\}) \in E(A)$, the statement follows.

By analogous arguments one shows that the results hold also for B, C and D, whenever $I_B \cap I_D \neq \emptyset$. If $I_B \cap I_D = \emptyset$ then we can conclude that $I_B = (I_C \cap I_B) \cup (I_D \cap I_B) = I_C \cap I_B$

and $I_D = (I_A \cap I_D) \cup (I_B \cap I_D) = I_A \cap I_D$. Hence, by definition of the vertex sets $V(BC)$ and $V(AD)$ together with Lemma 8.13 and 8.14 we obtain that $B \cong BC$ and $D \cong AD$ and thus, the assertion follows. $\square$

Theorem 8.19. *Connected, thin hypergraphs have a unique prime factor decomposition with respect to the normal product* $\breve{\boxtimes}$ *and the strong product* $\widehat{\boxtimes}$*, up to isomorphism and the order of the factors.*

Proof. For completeness, we will give the proof here, although it is essentially the same as the proof for graphs in [43, Lemma 5.38].

We proceed by induction w.r.t. the number of vertices. Therefore, let H be a connected thin hypergraph and assume the assertion is true for all hypergraphs having fewer vertices then H. Moreover, let

$$H_1 \boxtimes H_2 \boxtimes \ldots \boxtimes H_r = Q_1 \boxtimes Q_2 \boxtimes \ldots \boxtimes Q_s$$

be two prime factor decompositions of H. Then there are connected thin hypergraphs B and D, such that $H \cong H_1 \boxtimes B \cong Q_1 \boxtimes D$. Setting $H_1 \cong A$ and $Q_1 \cong C$, we have with Prop. 8.18, $H \cong AC \boxtimes AD \boxtimes BC \boxtimes BD$ and $H_1 \cong AC \boxtimes AD$ as well as $Q_1 \cong AC \boxtimes BC$. Since both H_1 and Q_1 are prime, it holds either $AC \cong K_1$ or $AD \cong BC \cong K_1$.

First, suppose AC is nontrivial. Hence, $AD \cong BC \cong K_1$, which implies $G_1 \cong AC \cong Q_1$. Furthermore, it follows $B \cong BC \boxtimes BD \cong BD \cong AD \boxtimes BD \cong D$. Since B resp. D have fewer vertices than H, we get $r = s$ and $H_i \cong Q_{\pi(i)}$ for $2 \leq i \leq r$ and some permutation π on $\{2, \ldots, r\}$.

Now, let $AC \cong K_1$. Then $H_1 \cong AD$ and $Q_1 \cong BC$. Furthermore

$$H_2 \boxtimes H_3 \boxtimes \ldots \boxtimes H_r \cong B \cong BC \boxtimes BD \cong Q_1 \boxtimes BD$$

and

$$Q_2 \boxtimes Q_3 \boxtimes \ldots \boxtimes Q_s \cong D \cong AD \boxtimes BD \cong H_1 \boxtimes BD.$$

W.l.o.g., we may assume $B \ncong D$. Since both B and D have fewer vertices than H, they have unique PFD. Hence $Q_1 \cong H_i$ for some $i \in \{2, \ldots, r\}$. W.l.o.g. suppose th notation to be chosen such that $Q_1 \cong H_2$. Then

$$D \cong Q_2 \boxtimes Q_3 \boxtimes \ldots \boxtimes Q_s \cong H_1 \cong BD \cong H_1 \boxtimes H_3 \boxtimes H_4 \boxtimes \ldots \boxtimes H_r.$$

Now, unique prime factorization follows immediately from the induction hypothesis, i.e., that BD has unique prime factorization and

$$H \cong H_1 \boxtimes B \cong H_1 \boxtimes Q_1 \boxtimes BD \cong Q_1 \boxtimes H_1 \boxtimes BD \cong Q_1 \boxtimes D \cong G.$$

$\square$

Thinness. We conclude this section by discussing the term "thinness". It is well-known that, although the PFD for a given graph G w.r.t. the strong graph product is unique, the coordinatizations might not be [28]. Therefore, the assignment of an edge being Cartesian or non-Cartesian is not unique in general. The reason for the non-unique coordinatizations is the existence of automorphisms that interchange vertices u and v, which is possible whenever u and v have the same neighborhoods and thus, if G is not thin. Thus, an important issue in the context of strong graph products is whether or not two vertices can be distinguished by their neighborhoods. The same holds for the normal and strong hypergraph product, as well. For graphs $G = (V, E)$, one defines the equivalence relation S on V with uSv iff $N^G[u] = N^G[v]$ and computes a so-called quotient graph G/S which is a thin graph. For this graph G/S the PFD is computed and one uses afterwards the knowledge of the *cardinalities* of the S-classes *only*, to find the prime factors of G. For graphs, one profits from the fact that all vertices $u_1, \ldots, u_n \in V(G)$ that share the same neighborhoods induce a complete subgraph K_n. Even in the proofs for the uniqueness results for the PFD w.r.t. the strong graph product of non-thin graphs, this fact is utilized. However, this technique cannot be used for hypergraphs in general, as the partial hypergraph formed by vertices that share the same neighborhoods need not be isomorphic, although the cardinalities of the S-classes might be the same. So far, we do not know, how to resolve this problem and state the following conjecture.

Conjecture 8.20. *Connected, simple, non-thin hypergraphs have a unique prime factor decomposition w.r.t.* $\breve{\boxtimes}$ *and* $\overset{\frown}{\boxtimes}$, *up to isomorphism and the order of the factors.*

8.3 Algorithms for the Construction of the Cartesian Skeleton and the Prime Factors

As shown by Bretto et al. [6] the PFD of hypergraphs with respect to the Cartesian product can be computed in polynomial time.

Theorem 8.21 ([6])**.** *The prime factors w.r.t. the Cartesian product of a given connected simple hypergraph* $H = (V, E)$ *with maximum degree* Δ *and rank* r *can be computed in* $O(|V||E|\Delta^6 r^6)$*, that is, in* $O(|V||E|)$ *time for hypergraphs* H *with a bounded rank and a bounded degree.*

The algorithm for computing the PFD of a given hypergraph with respect to the normal and the strong product works as follows. Analogously as for graphs, the key idea of finding the PFD with respect to $\boxtimes \in \{\breve{\boxtimes}, \overset{\frown}{\boxtimes}\}$ is to find the PFD of its Cartesian skeleton $\mathbb{S}(H)$ with respect to the Cartesian product and to construct the prime factors of H using the information of the PFD of $\mathbb{S}(H)$. In Algorithm 2 the pseudocode for determining the Cartesian skeleton $\mathbb{S}(H)$ is given. This Cartesian skeleton is afterwards factorized with the Algorithm of `Bretto et al.` [6] and one obtains the Cartesian prime factors of $\mathbb{S}(H)$. Note, for an arbitrary factorization $H = H_1 \boxtimes H_2$ of a thin hypergraph H, Proposition 8.9 asserts that $\mathbb{S}(H_1 \boxtimes H_2) = \mathbb{S}(H_1) \Box \mathbb{S}(H_2)$. Since $\mathbb{S}(H_i)$ is a spanning hypergraph of H_i, $i = 1, 2$, it follows that the $\mathbb{S}(H_i)$-layers of $\mathbb{S}(H_1) \Box \mathbb{S}(H_2)$ have the same vertex sets as the H_i-layers of $H_1 \boxtimes H_2$. Moreover, if $\boxtimes_{i \in I} H_i$ is the unique PFD of H then we have $\mathbb{S}(H) = \Box_{i \in I} \mathbb{S}(H_i)$. Since $\mathbb{S}(H_i)$, $i \in I$ need not be prime with respect to the Cartesian product, we can infer that the number of Cartesian prime factors of $\mathbb{S}(H)$, can be larger than the number of the strong or normal prime factors. Hence, given the PFD of $\mathbb{S}(H)$ it might be necessary to combine several Cartesian factors to get the strong or normal prime factors of H. These steps for computing the PFD with respect to $\boxtimes \in \{\breve{\boxtimes}, \widehat{\boxtimes}\}$ of a thin hypergraph are summarized in Algorithm 3.

For proving the time complexity of Algorithm 2 we need the following appealing result, established by Hammack and Imrich.

Lemma 8.22 ([27]). *For a given graph $G = (V, E)$ with maximum degree Δ the set of dispensable edges $D(H)$ and in particular, the Cartesian skeleton $\mathbb{S}(G)$ can be computed in $O(\min\{|E|^2, |E|\Delta^2\})$ time.*

Lemma 8.23. *For a given hypergraph $H = (V, E)$ with maximum degree Δ and rank r, Algorithm 2 computes the Cartesian skeleton $\mathbb{S}(H)$ in $O(|E|^2 r^4)$ time.*

Proof. The correctness of the algorithm follows immediately from Lemma 8.4.

For the time complexity observe that $[H]_2$ has at most $|E|\binom{r}{2}$ edges and that the maximum degree of $[H]_2$ is at most $\Delta(r-1)$. Hence, Lemma 8.22 implies that the computation of the set $D([H]_2)$ takes $O(\min\{|E|^2 r^4, |E| r^2 \Delta^2 r^2\}) = O(|E|^2 r^4)$ time. To check whether one of the at most $O(|E| r^2)$ pairs $\{x, y\} \in D([H]_2)$ is contained in one of the $|E|$ edges in H we need $O(|E|^2 r^2)$ time, from which we can conclude the statement. $\square$

For computing the time complexity of Algorithm 3 we first need the following lemma.

Lemma 8.24. *Let $H = (V, E)$ be a hypergraph with rank r and maximum degree Δ. Moreover, let $H_1, H_2 \subseteq H$ be partial hypergraphs of H such that $\mathbb{S}(H) \cong \mathbb{S}(H_1) \Box \mathbb{S}(H_2)$. The numbers $|\breve{\times}|$ and $|\widehat{\times}|$ of non-Cartesian edges in $H_1 \boxtimes H_2$, $\boxtimes \in \{\widehat{\boxtimes}, \breve{\boxtimes}\}$ can be computed in $O(r^2 + |V|\Delta^2)$ time.*

Proof. Let $H_1 = (V_1, E_1)$ and $H_2 = (V_2, E_2)$ be partial hypergraphs of H with rank r_1, resp., r_2 such that $\mathbb{S}(H) = \mathbb{S}(H_1) \Box \mathbb{S}(H_2)$. Note, it holds that $r_i \leq r$, $i = 1, 2$. For the cardinalities $|\breve{\times}|$ and $|\widehat{\times}|$ we have to compute for pairs of edges $e_1 \in E_1$ and $e_2 \in E_2$ several factorials and for the computation of the Stirling number we need in addition values of the form j^n. Note, that $m!$, resp., j^n can be computed in $O(1)$ time if one knows $(m-1)!$, resp., j^{n-1}. Hence, as preprocessing compute first the values $1, 2!, \ldots, r!$, which can be done in time complexity $O(r)$ and store them for later use. Analogously, the complexity for computing the values $j^2, \ldots, j^r$ for a fixed $j \in \{2, \ldots, r\}$ is $O(r)$. In that manner, we precompute and store the values $2^2, \ldots 2^r, \ldots, r^2, \ldots, r^r$ which takes $O(r^2)$ time. Finally, we store the values of the Stirling number, $S_{n,k}$ for $n = 1, \ldots, r$ and $k = 1, \ldots, r$. Note, $S_{n,k}$ can be computed in $O(1)$

Algorithm 2 `Cartesian Skeleton`

INPUT: A hypergraph $H = (V, E)$;
Compute the set $D([H]_2)$ of dispensable edges in $[H]_2$;
for every edge $\{x, y\} \in D([H]_2)$ **do**
  for all edges $e \in E$ with $x, y \in e$ remove e from E;
end for
OUTPUT: The partial hypergraph (V, E);

time, whenever $S_{n,k-1}$ is known. Hence, for $k, n = 1, \ldots, r$ the Stirling numbers $S_{n,k}$ can, together with the latter preprocessed stored values, be computed in $O(r^2)$ time. Therefore, these preprocessing steps have overall time complexity of $O(r^2)$.

After preprocessing and storing the latter mentioned values, one can compute the number of non-Cartesian edges in $e_1 \breve{\boxtimes} e_2$, resp., $e_1 \widehat{\boxtimes} e_2$ in $O(1)$ time, for a fixed pair $e_1 \in E_1$ and $e_2 \in E_2$. These computations are done for all pairs of edges $e_1 \in E_1$ and $e_2 \in E_2$. Hence, we have $|E_1||E_2|$ such computations to consider, which take altogether $O(|E_1||E_2|)$ time. Since $|E_i| \leq |V_i|\Delta_i$, $i = 1, 2$ we can conclude that $|E_1||E_2| \leq |V_1||V_2|\Delta_1\Delta_2$. Moreover, by definition of the products, it holds that $|V_1||V_2| = |V|$ and since $H_i \subseteq H$ we have $\Delta_i \leq \Delta$, $i = 1, 2$. Therefore, we end in an overall time complexity for computing $|\breve{\times}|$ and $|\widehat{\times}|$ of $O(r^2 + |V|\Delta^2)$. □

Theorem 8.25. *Algorithm 3 computes the prime factors w.r.t. $\boxtimes \in \{\widehat{\boxtimes}, \breve{\boxtimes}\}$ of a given thin connected simple hypergraph $H = (V, E)$ with maximum degree Δ and rank r in $O(|V||E|\Delta^6 r^6 + |V|^2|E|r)$ time.*

Proof. We start to prove the correctness of Algorithm 3. Since $H = (V, E)$ is thin, the Cartesian skeleton $\mathbb{S}(H)$ is uniquely determined and the Cartesian prime factors $H_i, i \in I$ of $\mathbb{S}(H)$ can be computed with the Algorithm of `Bretto et al.` [6]. This algorithm returns not only the prime factors of $\mathbb{S}(H)$ but also a coloring of the edges of $\mathbb{S}(H)$ and thus of the edges of H. That is, an edge $e \in E$ obtains color j if and only if $e \in E(\mathbb{S}(H))$ and e is an edge of some H_j-layer w.r.t. $\mathbb{S}(H) = \Box_{i \in I} H_i$. Hence, this colors the Cartesian edges of H w.r.t. the Cartesian PFD of $\mathbb{S}(H)$ and dispensable edges of H obtain no color. Based on $\mathbb{S}(H)$ one can compute the coordinates in the following way. One first computes $[\mathbb{S}(H)]_2$ and coordinatize the vertices of $V([\mathbb{S}(H)]_2) = V$ as proposed in [28, page 280] w.r.t. to the product coloring given by $\Box_{i \in I} H_i$. Note, then for all edges $e = \{x, y\} \in E([\mathbb{S}(H)]_2)$ holds $|p_i(e)| = 2$ if and only if the coordinates of x and y differ in the i-th coordinate and the other coordinates are identical. To prove that this is a valid coordinatization of $\mathbb{S}(H)$ one has to show, that for all edges $e \in E(\mathbb{S}(H))$ holds that $|p_i(e)| > 1$ if and only if for all $x, y \in e$ holds that x and y, differ in the i-th coordinate and the other coordinates are identical. Let $e \in E(\mathbb{S}(H))$ be an arbitrary edge. This edge forms a complete subgraph in the 2-section $[\mathbb{S}(H)]_2$. However, complete subgraphs must be contained entirely in one of the H_i-layers of

Algorithm 3 `PFD of thin hypergraphs w.r.t.` $\boxtimes \in \{\breve{\boxtimes}, \widehat{\boxtimes}\}$

1: **INPUT:** A thin hypergraph $H = (V, E)$;
2: Compute the Cartesian skeleton $\mathbb{S}(H)$ of H with Algorithm 2;
3: Compute the Cartesian PFD of $\mathbb{S}(H) = \Box_{i \in I} H_i$ by run of the algorithm of `Bretto et al.` [6]
4: Assign coordinates $c(v) = (c_1^v, \ldots c_{|I|}^v)$ w.r.t. $\Box_{i \in I} H_i$ to each vertex $v \in V$;
5: $J \leftarrow I$;
6: **for** $k = 1, \ldots, |I|$ **do**
7:     **for** each $S \subset J$ with $|S| = k$ **do**
8:         **for** $R \in \{S, I \setminus S\}$ **do**
9:             Compute $H^R \subseteq H$ with $V(H^R) = V(H)$ and $E(H^R) = \{e \in E(H) \mid |p_i(e)| = 1, i \in I \setminus R\}$;
10:         **end for**
11:         **if** all connected components of H^S, resp., $H^{I \setminus S}$ are isomorphic **then**
12:             take one connected component H_S of H^S, resp., $H_{I \setminus S}$ of $H^{I \setminus S}$;
13:             **if** all non-Cartesian edges w.r.t. the factorization $H_S \boxtimes H_{I \setminus S}$ are contained in H **then**
14:                 save H_S as prime factor;
15:             **end if**
16:         **end if**
17:     **end for**
18: **end for**
19: **OUTPUT:** The prime factors of H;

$[\mathbb{S}(H)]_2$, as complete graphs are so-called S-prime graphs, see e.g. [32, 38]. From this we can conclude that the computed coordinates of vertices in $[\mathbb{S}(H)]_2$ give a valid coordinatization of the vertices in $\mathbb{S}(H)$.

Now, consider Line 6-18. We finally have to examine which "combination" of the proposed Cartesian prime factors are prime factors w.r.t. $\boxtimes$ (Line 6-18). For this, we search for the minimal subsets S of I such that the subgraph H_S and $H_{I \setminus S}$, where H_S is one connected component of H^S and $H_{I \setminus S}$ is one connected component of $H^{I \setminus S}$, correspond to layers of a factor of H w.r.t. $H_S \boxtimes H_{I \setminus S}$. We continue to check whether all connected components of H^S, resp., $H^{I \setminus S}$ are isomorphic and if so, we test whether all non-Cartesian edges w.r.t. the factorization $H_S \boxtimes H_{I \setminus S}$ are present. If this is the case, H_S is saved as prime factor of H w.r.t. $\boxtimes$. Reasoning exactly as in the proof for graphs in [28, Chapter 24.3] together with the preceding results, we conclude the correctness of this part in Line 6-18.

We are now concerned with the time complexity. Note, since we assumed the hypergraph $H = (V, E)$ to be connected we can conclude that $[H]_2$ has at least $|V| - 1$ edges. Moreover, the number of edges in $[H]_2$ does not exceed $|E|r^2$ and therefore we can conclude that

$O(|V|^2) \subseteq O(|V||E|r^2)$. Furthermore, we will make in addition frequent use of the fact that $|E| \leq |V|\Delta$. Now, consider Line 2-4. Lemma 8.23 implies that the Cartesian skeleton can be computed in $O(|E|^2r^4) \subseteq O(|V|^2\Delta^2r^4) \subseteq O(|V||E|\Delta^2r^6)$ time and by Theorem 8.21 we have that the PFD of $\mathbb{S}(H)$ can be computed in $O(|V||E|\Delta^6r^6)$ time. For the computation of the coordinates we use the 2-section $[\mathbb{S}(H)]_2$ as described in the previous part of this proof. Note, $[\mathbb{S}(H)]_2$ has at most $|E|r^2$ edges and the coordinates can therefore be computed in $O(|E|r^2)$, see [28, Chapter 23.3]. Hence, the overall time complexity of the steps in Line 2-4 is $O(|V||E|\Delta^6r^6)$.

Consider now Line 6-18. Clearly, each H^R can be computed in $O(|E|r)$ time. For finding the connected components of H^R in Line 11 one can use its 2-section $[H^R]_2 = (V, E')$ and apply the classical breadth-first search to it, which has time complexity $O(|E'| + |V|)$. Let Δ' be the maximum degree of $[H^R]_2$ which is bounded by Δr. Hence, we can determine the connected components of H^R in time complexity $O(|E'| + |V|) \subseteq O(|V|\Delta') \subseteq O(|V|\Delta r)$. Moreover, in Line 11 we have to perform an isomorphism test for a fixed bijection given by the coordinates which takes $O(|E|r)$ time. This test must be done for each of the connected components of H^R which are at most $|V|$. Hence, the latter task has time complexity $O(|V||E|r)$. Taken together the preceding considerations and since $\Delta \leq |E|$ we can conclude that Line 11 can be performed in $O(|V|\Delta r + |V||E|r) = O(|V||E|r)$ time. To test whether all non-Cartesian edges w.r.t. $H_S \boxtimes H_{I\setminus S}$ are contained in H (Line 13) we examine whether putative non-Cartesian edges $e \in E(H) \setminus E(H_S \Box H_{I\setminus S})$ are valid non-Cartesian edges, that is, we prove if the projection properties for these edges into the factors fulfill the condition (ii) in the definition of edges in $H_S \boxtimes H_{I\setminus S}$ and count them, if valid. If the counted number is identical to $|\breve{\times}|$, resp., $|\widehat{\times}|$ we are done. Since the coordinates are given, the projections can be computed in $O(|E|r)$ time. The computation of $|\breve{\times}|$, resp., $|\widehat{\times}|$ has time complexity $O(r^2 + |V|\Delta^2)$ (Lemma 8.24). Thus, Line 13 can be performed in $O(|E|r + r^2 + |V|\Delta^2)$ time. Taken together all the single tasks in Line 8-16 we end up in a time complexity $O(|E|r + |V||E|r + |V|\Delta^2 + r^2) = O(|V||E|r + |V|\Delta^2 + r^2)$. Assume all these tasks are done for each of the the $2^{|I|}$ subsets of I. Since $|I|$ is the number of factors of $\mathbb{S}(H)$ and thus, is bounded by $\log_2(|V|)$ we have at most $|V|$ subsets of I. To summarize, the total complexity of Line 6-18 is $O(|V|^2|E|r + |V|^2\Delta^2 + |V|r^2)$. Since H is assumed to be connected we can conclude that $O(|V|^2) \subseteq O(|V||E|r^2)$ and hence, the complexity of Line 6-18 is $O(|V|^2|E|r + |V||E|\Delta^2r^2 + |V|r^2)$.

Taken together the preceding results we can infer that Algorithm 3 has time complexity $O(|V||E|\Delta^6r^6 + |V|^2|E|r + |V||E|\Delta^2r^2 + |V|r^2)$, that is, $O(|V||E|\Delta^6r^6 + |V|^2|E|r)$. $\square$

Corollary 8.26. *Algorithm 3 computes the prime factors w.r.t. $\boxtimes \in \{\widehat{\boxtimes}, \breve{\boxtimes}\}$ of a given thin connected simple hypergraph $H = (V, E)$ with bounded degree and bounded rank in $O(|V|^2|E|)$ time.*

Chapter 9

Summary and Outlook

In this thesis, we considered graphs and hypergraphs that have (relaxed) product structures.

We discussed in detail *RSP-relations*, a relaxation of relations fulfilling the square property and therefore of the product relation σ. As it turned out, such relations are hard to handle in graphs that contain $K_{2,3}$-subgraphs. On the other hand, it is possible to determine finest RSP-relations in polynomial time in $K_{2,3}$-free graphs. Explicit constructions of such relations in complete and complete bipartite graphs were given.

Furthermore, we established the close connection of (*well-behaved*) RSP-relations to (quasi-)covers of graphs and equitable partitions. Thereby, we characterized the existence of non-trivial RSP-relations by means of the existence of spanning subgraphs that yield quasi-covers of the graph under investigation. We found out that all layers of a graph w.r.t an equivalence class of a well-behaved RSP-relation share a common cover, and how such a covering graph can be constructed via this relation. It was shown that for any equivalence class φ of a well-behaved RSP-relation R the connected components of the graph $G_{\overline{\varphi}} = (V(G), E(G) \setminus \varphi)$ form a natural equitable partition $\mathcal{P}^R_{\overline{\varphi}}$ of the vertex set of $G_\varphi = (V(G), \varphi)$. Moreover, the common refinement $\mathcal{P}^R$ of these partitions $\mathcal{P}^R_{\overline{\varphi}}$ yields again an equitable partition of $V(G)$ and the quotient $G/\mathcal{P}^R$ has then a product representation as $G/\mathcal{P}^R \cong \Box_{\varphi \sqsubseteq R} G_\varphi / \mathcal{P}^R_{\overline{\varphi}}$. This product structure of the quotient graph is still retained even for RSP-relations that are not well-behaved. In addition, it was shown that coarse grainings of the respective relation on the edge set may lead to refinements of the vertex partition.

We determined (finest) RSP-relations in certain graph products from given (finest) RSP-relations of the factors and showed in what manner the quotient graphs of the product w.r.t such an RSP-relation result from the quotient graphs of the factors and the respective product. As it turned out, in case of the strong product, the quotient constructed from its factors collapses to the one vertex graph with a loop. In case of Cartesian product, computing quotients w.r.t. those relations and multiplying graphs commutes.

In addition, we examined relations on the edge sets of *hyper*graphs that satisfy the grid property, the hypergraph analog of the square property. We introduced the *strong* and the *relaxed* grid property as variations of the grid property, the latter generalizing the relaxed square property. We thereby showed, that many, although not all results for graphs and the (relaxed) square property can be transferred to hypergraphs. For instance, we showed that similar to the graph case, any equivalence relation R on the edge set of a hypergraph H that satisfies the relaxed grid property induces partitions P^R_{φ}, $\varphi \sqsubseteq R$ and P^R of $V(H)$ such that the quotient $H/\mathcal{P}^R$ is then a Cartesian product, $H/\mathcal{P}^R \cong \Box_{\varphi \sqsubseteq R} H_{\varphi}/\mathcal{P}^R_{\varphi}$. Besides, we introduced the notion of *(Cartesian) hypergraph bundles*, the analog of (Cartesian) graph bundles. We showed, that any 2-convex equivalence relation on the edge set of a hypergraph that satisfies the grid property, yield the structural properties of a hypergraph bundle, which generalizes a well-known result for graph bundles.

Finally, we showed that every connected thin hypergraph H has a unique prime factorization with respect to the normal and strong (hypergraph) product. Both products coincide with the usual strong *graph* product whenever H is a graph. We introduced the notion of the Cartesian skeleton of hypergraphs as a natural generalization of the Cartesian skeleton of graphs and proved that it is uniquely defined for thin hypergraphs. Moreover, we showed that the Cartesian skeleton of hypergraphs can be determined in $O(|E|^2)$ time and that the PFD can be computed in $O(|V|^2|E|)$ time, for hypergraphs $H = (V, E)$ with bounded degree and bounded rank.

Still, many interesting problems remain open topics for further research. Concerning RSP-relations, it would be worth to determine the complexity of the problem of determining finest (well-behaved) RSP-relations. Since there is a close connection to graph covers, we suspect that the latter problem is NP-hard. If so, then fast heuristics need to be designed. It is also of interest to investigate, for which graph classes (that are more general than $K_{2,3}$-free graphs) the proposed algorithm determines well-behaved or finest RSP-relations. Moreover, one might ask, under which circumstances is it possible to guarantee that there is a non-trivial finest RSP-relation that is in addition well-behaved. Note, the graph $G = K_{2,3}$ has no such relation. However, there might be interesting graph classes that have one. In addition, it might be of particular importance (also for computational aspects) to distinguish RSP-relations. Let us say that two RSP-relations R and S on E are *equivalent*, $R \cong S$, if there is an automorphism $f : V \to V$ such that $([x, y], [a, b]) \in R$ if and only if $([f(x), f(y)], [f(a), f(b)]) \in S$. Note, if $G = K_{2,3}$ then all finest RSP-relations consist of two equivalence classes and all such relations are equivalent. Clearly, if $R \cong S$, then $G/\mathcal{P}^R \cong G/\mathcal{P}^S$. However, the converse is not true, i.e., $G/\mathcal{P}^R \cong G/\mathcal{P}^S$ does not imply $R \cong S$, see Example 3.21. This suggests to consider under which conditions finest RSP-relations are unique, or for which graphs the equivalence of RSP-relations can be expressed in terms of isomorphism of quotient graphs.

Furthermore, it was shown that coarse grainings of the respective relation on the edge set of a graph may lead to refinements of the vertex partition. However, it is not yet resolved definitively when these refinements are strict.

At this point, not all results concerning graphs and RSP-relations could be generalized to hypergraphs. As an example, for equitable partitions, there is no counterpart on hypergraphs defined yet. Since, in contrast to simple graphs, the number of neighbors need not coincide with the number of edges, there may also be various ways to extend equitable partitions to hypergraphs, also with regard to locally bijective homomorphisms. One could then examine, which kind of partition is induced by which variation of the grid property.

Regarding hypergraph bundles, a complete characerization in terms of varied grid properties is still lacking. Recall that in the graph case any 2-convex equivalence relation on the edge set of a graph G satisfying the unique square property induces a fundamental facorization of G as a graph bundle. Conversely any partition of the edges of a graph bundle with respect to such a fundamental factorization has the unique square property.

Beyond this, one could also build up relaxed product structures for graphs and hypergraphs by less restrictive rules then those for building Cartesian product (hyper)graphs from the factors. (Hyper)graph bundles serve as an example for those constructions. They are build up from a base (hyper)graph B and a fiber F by assigning each edge of B an automorphism of F and then connecting vertices of the vertex set $V(B) \times V(F)$ by certain rules. One could now consider more general structures, e.g., by replacing $Aut(F)$ with larger groups acting on $V(F)$. As an example, consider the following construction: For a graph F, denote with $Cov(F)$ the set of covering projections from F. Let B be another graph and define a mapping $\pi : V(B) \to Cov(F)$. That is, each vertex $b \in V(B)$ is assigned a covering projection $\pi(b) : F \to F'$ for some F' that is covered by F. For brevity, we will write π_b instead of $\pi(b)$. Note, F' is already uniquely determined by π_b, namely, $F' = \pi_b(F)$. We now define the graph G as follows:

(1) $V(G) = \dot{\bigcup}_{b\in V(B)} V(\pi_b(F)) = \{\pi_b(v) \mid b \in V(B) \text{ and } v \in V(F)\}$

(2) two vertices $\pi_{b_1}(v_1)$ and $\pi_{b_2}(v_2)$ are adjacent in G if one of the following conditions is satisfied:

  (2i) $b_1 = b_2$ and $[v_1, v_2] \in E(F)$, or

  (2ii) $[b_1, b_2] \in E(B)$ and $v_1 = v_2$.

We will call G a *relaxed graph bundle*.

It is easy to see, that the equivalence relation R on $E(G)$ whose classes are edges of type (2i) and those of (2ii) is an RSP-relation. The task is to find a complete characterization of those structures, similar to the case of graph bundles, in terms of RSP-relations that may satisfy additional properties. Moreover, one could ask if there exists an equivalent definition of those structures via graph maps and local constraints. Another interesting question would be which invariants propagate under this construction from the graphs F and B to G. Subsequently, one could also examine structures where $Cov(F)$ is replaced by $Hom(F)$.

Finally, it is not known yet if non-thin hypergraphs enjoy unique prime factorization w.r.t. the strong and the normal product. For non-thin hypergraphs, the Cartesian skeleton might not be uniquely determined, as it is not unique for non-thin graphs. In the graph case, this problem is solved by computing a thin graph from a given graph that collapses vertices with identical neighborhood, so-called S-classes, to one vertex. It then suffices to know the cardinalities of these S-classes for computing the PFD of the original graph. However, for hypergraphs this approach may fail, since, in contrast to the graph case, the same number of neighbors does not imply local isomorphisms. This problem would also be worth to resolve it.

APPENDIX

Appendix A

Proofs

Proof of Lemma 3.19. At first we prove that R is an equivalence relation. That is, we have to show that $\varphi_i \cap \varphi_j = \emptyset$ for all $i \neq j$ and $E(K_m) = \bigcup_{i=1}^{l} \varphi_i$. For contraposition suppose, $\varphi_i \cap \varphi_j \neq \emptyset$ for some $i \neq j$. That is, there exists $x, y \in V(K_m) = \{0 \dots, m-1\}$ such that $[x, (x+i) \mod m] = [y, (y+j) \mod m]$. Notice, $x+i < 2m$ as well as $y+j < 2m$. Thus, we have $x+i = p \cdot m + (x+i) \mod m$ and $y+j = q \cdot m + (y+j) \mod m$ with $p, q \in \{0, 1\}$. First assume $x = y$. Hence, $(x+i) \mod m = (x+j) \mod m$ and we obtain $|i-j| = |p-q| \cdot m$ with $|q-p| \in \{0, 1\}$. If $|p-q| = 0$ it follows $i = j$. Therefore suppose, $|p-q| = 1$. This implies $|i-j| = m \geq 2l$ and moreover, $|i-j| < l$ since $i, j \in \{1, \dots, l\}$, a contradiction.

Now, assume $x \neq y$. Then it must hold $x = (y+j) \mod m$ and $y = (x+i) \mod m$ if $[x, (x+i) \mod m] = [y, (y+j) \mod m]$. Hence, with our considerations above, we get $i + j = (p+q) \cdot m$ with $p+q \in \{0, 1, 2\}$. From $i, j \in \{1, \dots, l\}$, we conclude $0 < i+j \leq 2l$ which implies in particular $p+q > 0$. It follows $2l \leq m \leq i+j \leq 2l$, hence $i = j = l$ which contradicts the choice of i, j. Thus, $\varphi_i \cap \varphi_j = \emptyset$ for all $i, j \in \{1, \dots, l\}$ with $i \neq j$.

Next, we show $|\bigcup_{i=1}^{l} \varphi_i| = |E(K_m)|$. Since $\varphi_i \subseteq E(K_m)$ for all $i \in \{1, \dots, l\}$, we then can conclude $\bigcup_{i=1}^{l} \varphi_i = E(K_m)$. First, let $i < \frac{m}{2}$. Assume, there exists $x \in \{0, \dots, m-1\}$ such that $x = (x+i) \mod m$. From previous considerations, it follows $i = p \cdot m$ with $p \in \{0, 1\}$, which contradicts $0 < 1 \leq i \leq l < m$. Now suppose, there are $x, y \in \{0, \dots, m-1\}$ such that $[x, (x+i) \mod m] = [y, (y+i) \mod m]$. If $x \neq y$, it follows $x = (y+i) \mod m$ and $y = (x+i) \mod m$. As before, we conclude $2i = (p+q) \cdot m$ with $p+q \in \{0, 1, 2\}$ and since $i > 0$, we have $p+q > 0$. Thus, $m \leq 2i < m$, which is a contradiction. Hence, $|\varphi_i| = |\{0, \dots, m-1\}| = m$ for all $i < \frac{m}{2}$. If $i = \frac{m}{2}$, and thus, m is even, we have $|\varphi_{\frac{m}{2}}| = \frac{m}{2}$, since for all $x < \frac{m}{2}$ it holds that $[x, x+\frac{m}{2}] = [x+\frac{m}{2}, (x+\frac{m}{2}+\frac{m}{2}) \mod m]$. It follows $|\bigcup_{i=1}^{l} \varphi_i| = \sum_{i=1}^{l} |\varphi_i| = l \cdot m = \frac{(m-1) \cdot m}{2} = |E(K_m)|$ if m is odd and $|\bigcup_{i=1}^{l} \varphi_i| = \sum_{i=1}^{l-1} |\varphi_i| + |\varphi_{\frac{m}{2}}| = (l-1) \cdot m + \frac{m}{2} = \frac{(m-1) \cdot m}{2} = |E(K_m)|$ if m is even. Therefore, R is an equivalence relation on $E(K_m)$.

It remains to show that R has the relaxed square property and there is no refinement of

R with this property. Therefore, let $e = [x, y] \in \varphi_i$ and $f = [x, z] \in \varphi_j$, $i \neq j$. We have to show, that there exists a vertex $w \in V(K_m)$ such that $[y, w] \in \varphi_j$ and $[z, w] \in \varphi_i$. $[x, y] \in \varphi_i$ implies $y = (x+i) \mod m$ or $x = (y+i) \mod m$ and $[x, z] \in \varphi_j$ implies $z = (x+j) \mod m$ or $x = (z+j) \mod m$. If $y = (x+i) \mod m$ and $z = (x+j) \mod m$, we choose $w = (y+j) \mod m$. It is clear, that $w \neq x, y, z$. By definition, it holds that $[y, w] \in \varphi_j$. Moreover, by simple calculation we get with the preceding $w = (z + i) \mod m$ and hence $[z, w] \in \varphi_i$. If $y = (x+i) \mod m$ and $x = (z+j) \mod m$, we choose $w = (z+i) \mod m$, then $w \neq x, y, z$. Hence, $[w, z] \in \varphi_i$. In this case we get $y = (w + j) \mod m$ that is $[y, w] \in \varphi_j$. If $x = (y + i) \mod m$ and $z = (x + j) \mod m$, we choose $w = (y + j) \mod m$. Again $w \neq x, y, z$, and by definition $[y, w] \in \varphi_j$. Here, we obtain $z = (w + i) \mod m$ and hence $[z, w] \in \varphi_i$. If $x = (y + i) \mod m$ and $x = (z + j) \mod m$, we choose w such that $z = (w + i) \mod m$, that is $[z, w] \in \varphi_i$. In this case we have $w \neq x, y, z$ and moreover, $y = (w + j) \mod m$ and hence $[y, w] \in \varphi_j$. That is, R has the relaxed square property.

We show now, that no equivalence class φ of R can be split into two classes $\varphi_i = \psi_{i_1} \cup \psi_{i_2}$, such that the equivalence relation S, that has classes $\varphi_1, \ldots, \varphi_{i-1}, \psi_{i_1}, \psi_{i_2}, \varphi_{i+1}, \ldots, \varphi_l$ is an RSP-relation. Therefore, notice that each vertex $x \in V(K_m)$ is incident to exactly two φ_i edges for all $i < \frac{m}{2}$, namely $[x, (x + i) \mod m]$ and $[x, (x - i) \mod m]$, thus the layers are all cycles for $i < \frac{m}{2}$. Moreover, each vertex $x \in V(K_m)$ is incident to exactly one $\varphi_{\frac{m}{2}}$-edge. Recalling Lemma 3.5, $\varphi_{\frac{m}{2}}$ cannot be split. For $k < \frac{m}{2}$ let C be the φ_k-layer containing vertex 0. It has edges $[0, k], [k, 2k], [2k, 3k \mod m], \ldots, [(q - 1) \cdot k, 0]$ with $q \cdot k \mod m = 0$. By Lemma 3.6, any edge in C must be contained in a square, hence C itself must be a square and thus has edges $[0, k], [k, 2k], [2k, 3k], [3k, 0]$ with $4k = m$, since $k < \frac{m}{2}$ and $k > 1$ since $m \neq 4$. Because S is an RSP-relation, it holds that $([0, k], [2k, 3k]), ([k, 2k], [3k, 0]) \in S$ and $([0, k], [k, 2k]), ([2k, 3k], [3k, 0]) \notin S$ by Lemma 3.5. Consider the edges $[0, k] \in \varphi_k$ and $[0, 1] \in \varphi_1 \neq \varphi_k$, hence they are in different S-classes. Vertex $k \in V(K_m)$ is incident to exactly two φ_1-edges, namely $[k, k + 1]$ and $[k, k - 1]$. Since $[1, k - 1] \in \varphi_{k-2} \neq \varphi_k$, the only possible square spanned by $[0, k]$ and $[0, 1]$ with opposite edges in the same S-class is $0 - 1 - (k+1) - k$ with $[0, k], [k, k+1] \in S$. Now, consider edges $[k, 2k] \in \varphi_k$ and $[1, k] \in \varphi_{k-1}$. Vertex $2k \in V(K_m)$ is incident to exactly two φ_{k-1}-edges, namely $[2k, k + 1]$ and $[2k, 3k - 1]$. Since $[1, 3k - 1] \in \varphi_{k+2} \neq \varphi_k$, the only possible square spanned by $[k, 2k]$ and $[1, k]$ with opposite edges in the same S-class is $1 - k - 2k - (k + 1)$ with $([1, k + 1], [k, 2k]) \in S$. Thus, $([0, k], [k, 2k]) \in S$, a contradiction. Hence, R is finest RSP-relation on K_m for all $m \neq 4$. $\square$

Proof of Lemma 3.20. It is easy to check that R is an RSP-relation. It remains to show, that it is a finest one. By Lemma 3.11, the equivalence class φ cannot be split into two equivalence classes since the vertex 0 is incident with only one edge of φ. On the other hand, every vertex in $\{2, \ldots, m - 1\}$ is incident with exactly two edges in $\overline{\varphi}$, therefore if $\overline{\varphi}$ can be split into two equivalence classes, the edges $[0, 2]$ and $[1, 2]$ must be in different equivalence classes. The definition of RSP-relations implies that $[0, 2]$ and $[2, 3]$ must lie on a common square with opposite edges in the same equivalence class. The only possible candidate is the square $0-2-3-1$, thus $[0,2]$ and $[1,3]$ must be in the same class. Similarly, $[1,3]$ and $[3,4]$ must lie on a common square with opposite edges in the same equivalence class. The only possible candidate is the square $1-3-4-0$, thus $[1,3]$ and $[0,4]$ must be in the same class. Now, we use the same arguments for edges $[0,4]$ and $[4,2]$ to find out that $[0,4]$ and $[1,2]$ are in the same class. Since the relation is transitive $[0,2]$ and $[1,2]$ must be in the same class, a contradiction with the assumption that $\overline{\varphi}$ can be split. □

Proof of Lemma 3.22. Notice, that with our notation we have $E(K_{m,m}) = E(K_2 \boxtimes K_m) \setminus (E(K_m^x) \cup E(K_m^y))$ with $x, y \in V(K_2) \times V(K_m)$ s.t. $p_1(x) \neq p_1(y)$. With Lemma 5.11 and Lemma 3.6, it follows that R is an RSP-relation on $E(K_{m,m})$. It is clear that any refinement of S leads to a refinement of R. Thus we just have to show the converse, i.e., that R is a finest RSP-relation if S is finest RSP-relation. Let φ denote the equivalence class defined by condition (1), i.e., $\varphi = \{e \in E(K_{m,m}) \mid |p_2(e)| = 1\}$. By construction, each vertex is adjacent to exactly one φ-edge, therefore, φ cannot be split by Lemma 3.5. Moreover, two adjacent edges e, f with $e \in \varphi$ and $f \in \psi \neq \varphi \sqsubseteq R$ span exactly one square with opposite edges in the same equivalence classes, namely the square with $p_2(f) = p_2(f')$, where f' is opposite edge of f. Therefore, $p_2(e) = p_2(e')$ implies $(e, e') \in Q$ for any refinement Q of R with relaxed square property. Furthermore, with our notations, any refinement Q of R leads also to a refinement $Q_{|E(K_2 \times K_m)}$ of $R_{|E(K_2 \times K_m)}$, the restrictions of Q and R to $E(K_2 \times K_m) \subseteq E(K_{m,m})$, respectively. If the refinement Q is proper and satisfies the relaxed square property on $E(K_{m,m})$, the same is true for $Q_{|E(K_2 \times K_m)}$ on $E(K_2 \times K_m)$ by Lemma 3.6 and our previous considerations. Moreover, we can conclude that Q determines an equivalence relation $p_2(Q)$ on K_m via $(p_2(e), p_2(f)) \in p_2(Q)$ iff $(e, f) \in Q$. It holds $p_2(C_4) \cong C_4$ for any square in $K_2 \times K_m$. Furthermore, $p_2(e) = p_2(e')$ implies $(e, e') \in Q$ if Q has the relaxed square property. Therefore, it follows, $p_2(Q)$ is a proper refinement of S with the relaxed square property if Q is a proper refinement of R with the relaxed square property. This completes the proof. □

Proof of Lemma 3.23. It is clear, that R is an equivalence relation. Thus, it remains to show that R has the relaxed square property. Therefore, let $e, f \in E(K_{m,n})$ such that $(e, f) \notin R$. Notice, by construction it holds that $\psi' \neq \varphi'$ if and only if $\psi \neq \varphi$ for all $\psi', \varphi' \sqsubseteq S$ and $\psi, \varphi \sqsubseteq R$ with $\psi' \subseteq \psi$ and $\varphi' \subseteq \varphi$.

First, suppose that e and f are incident in some vertex $y_r \in V(K_{m,n})$, $r \in \{1, \ldots, n\}$. That is, $e = [x_j, y_r]$ and $f = [x_l, y_r]$ for some $j, l \in \{1, \ldots, m\}, j \neq l$. If $r \leq m$ then by construction $e, f \in E(K_{m,m})$ and $(e, f) \notin S$, and hence they span a square with opposite edges in the same equivalence classes of S, which is also retained in $K_{m,n}$ with the same properties. If $r > m$, then $r = m + i$ for some $i \in \{1, \ldots, n-m\}$. By construction, there exists $k_i \in \{1, \ldots, m\}$ such that $([x_j, y_{k_i}], [x_j, y_{m+i}]) \in R$ and $([x_l, y_{k_i}], [x_l, y_{m+i}]) \in R$, which implies $([x_j, y_{k_i}], [x_l, y_{k_i}]) \notin R$ and hence, by construction, $([x_j, y_{k_i}], [x_l, y_{k_i}]) \notin S$. Since S has the relaxed square property, there exists $w \in V(K_{m,m}) \subset V(K_{m,n})$ such that $[x_j, y_{k_i}]$

and $[x_l, y_{k_i}]$ span a square $x_j - y_{k_i} - x_l - w$, such that $([x_l, w], [x_j, y_{k_i}]) \in S \subset R$ and $([x_j, w], [x_l, y_{k_i}]) \in S \subset R$. Then $x_j - y_{m+i} - x_l - w$ is a square spanned by e and f with opposite edges in the same equivalence class.

Now assume e and f are incident in some vertex $x_j \in V(K_{m,n})$, $j \in \{1, \dots, m\}$. That is, $e = [x_j, y_r]$ and $f = [x_j, y_s]$ for some $r, s \in \{1, \dots, n\}, r \neq s$. If $r, s \leq m$, then by construction $e, f \in E(K_{m,m})$ and $(e, f) \notin S$, and hence they span a square with opposite edges in the same equivalence classes of S, which is also retained in $K_{m,n}$ with the same properties. If $r, s > m$, then $r = m + i$, $s = m + l$ for some $i, l \in \{1, \dots, n - m\}$. By construction, there exists $k_i, k_l \in \{1, \dots, m\}$ such that $([x_j, y_{m+i}], [x_j, y_{k_i}]) \in R$ as well as $([x_j, y_{m+l}], [x_j, y_{k_l}]) \in R$, from which we can conclude $([x_j, y_{k_i}], [x_j, y_{k_l}]) \notin R$. By construction we have $([x_j, y_{k_i}], [x_j, y_{k_l}]) \notin S$, and since S has the relaxed square property, there exists $w \in V(K_{m,m}) \subset V(K_{m,n})$ such that $[x_j, y_{k_i}]$ and $[x_j, y_{k_l}]$ span a square $(x_j, y_{k_i}, w, y_{k_l})$, such that $([w, y_{k_l}], [x_j, y_{k_i}]) \in S \subset R$ and $([w, y_{k_i}], [x_j, y_{k_l}]) \in S \subset R$. Moreover, by construction, we have $([w, y_{m+i}], [w, y_{k_i}]) \in R$ as well as $([w, y_{m+l}], [w, y_{k_l}]) \in R$. Thus $x_j - y_{m+i} - w - y_{m+l}$ is a square spanned by e and f with opposite edges in the same equivalence class. If $r > m, s \leq m$, then $r = m+i$ for some $i \in \{1, \dots, n-m\}$. By construction, there exists $k_i \in \{1, \dots, m\}$ such that $([x_j, y_{m+i}], [x_j, y_{k_i}]) \in R$ and thus, $([x_j, y_{k_i}], [x_j, y_l]) \notin R$, hence, $([x_j, y_{k_i}], [x_j, y_{k_l}]) \notin S$. Since S has the relaxed square property, there exists $w \in V(K_{m,m}) \subset V(K_{m,n})$ such that $[x_j, y_{k_i}]$ and $[x_j, y_l]$ span a square $x_j - y_{k_i} - w - y_l$, such that $([w, y_l], [x_j, y_{k_i}]) \in S \subset R$ and $([w, y_{k_i}], [x_j, y_l]) \in S \subset R$. Moreover, by construction, we have $([w, y_{m+i}], [w, y_{k_i}]) \in R$. Hence, $x_j - y_{m+i} - w - y_l$ is a square spanned by e and f with opposite edges in the same equivalence class. Analogously, one shows that e and f span a square with opposite edges in the same equivalence class if $r \leq m$ and $s > m$, which completes the proof. $\square$

Proof of Lemma 6.4. To prove validity of the formula for $|\breve{\times}|$, we show that e is a non-Cartesian edge in $H_1 \breve{\boxtimes} H_2$ if and only if there are edges $e_1 \in E(H_1)$ and $e_2 \in E(H_2)$ such that $p_1(x) \mapsto p_2(x)$ for all $x \in e$ defines an injective mapping $e_1 \to e_2$ whenever $|e_1| \leq |e_2|$ and else that $p_2(x) \mapsto p_1(x)$ for all $x \in e$ defines an injective mapping $e_2 \to e_1$.

Let e be a non-Cartesian edge in $H_1 \breve{\boxtimes} H_2$. Clearly, by definition of the normal product, there are edges $e_1 \in E(H_1)$ and $e_2 \in E(H_2)$ with $e \in E(e_1 \breve{\boxtimes} e_2)$. Assume w.l.o.g. $|e_1| \leq |e_2|$, otherwise interchange the role of e_1 and e_2. By definition of the normal product it holds $|p_1(e)| = |p_2(e)| = |e| = |e_1| \leq |e_2|$. Thus, we have $p_1(e) = e_1 \in E(H_1)$. Therefore, we can conclude that all vertices of e differ in each coordinate, and thus, $p_1(x) \neq p_1(x')$ implies $p_2(x) \neq p_2(x')$ for all distinct vertices $x, x' \in e$. Since $p_2(e) \subseteq e_2$, it follows that $p_1(x) \mapsto p_2(x)$, $x \in e$ indeed defines an injective mapping $e_1 \to e_2$. Conversely, if there are edges $e_1 \in E(H_1)$ and $e_2 \in E(H_2)$ such that w.l.o.g. $p_1(x) \mapsto p_2(x)$, $x \in e$ defines an injective mapping $e_1 \to e_2$, we can conclude that $p_1(e) = e_1$ and $p_2(e) \subseteq e_2$. Since $p_1(x) \mapsto p_2(x)$, $x \in e$ is a mapping, we have $|e| = |e_1|$ and by injectivity, it follows $|e_1| = |p_1(e)| = |p_2(e)| \leq |e_2|$. Hence, e satisfies the condition *(ii)* in the definition of the edges in the normal product and thus, $e \in E(H_1 \breve{\boxtimes} H_2)$. Finally, it is well-known, that for any two sets N, M with $|N| \leq |M|$

there are $\frac{|M|!}{(|M|-|N|)!}$ injective mappings from N to M. Applying this result to every pair of edges $e_1 \in E(H_1)$ and $e_2 \in E(H_2)$ the assertion for $|\overset{\smile}{\times}|$ follows.

To prove validity of the formula for $|\overset{\frown}{\times}|$, we show that e is a non-Cartesian edge in $H_1 \mathbin{\widehat{\boxtimes}} H_2$ if and only if there are edges $e_1 \in E(H_1)$ and $e_2 \in E(H_2)$ such that $p_1(x) \mapsto p_2(x)$ for all $x \in e$ defines a surjective mapping $e_1 \to e_2$ whenever $|e_1| \geq |e_2|$ and else that $p_2(x) \mapsto p_1(x)$ for all $x \in e$ defines a surjective mapping $e_2 \to e_1$.

Let e be a non-Cartesian edge in $H_1 \mathbin{\widehat{\boxtimes}} H_2$. Clearly, by definition of the strong product, there are edges $e_1 \in E(H_1)$ and $e_2 \in E(H_2)$ with $e \in E(e_1 \mathbin{\widehat{\boxtimes}} e_2)$. Assume w.l.o.g. $|e_1| \geq |e_2|$, otherwise interchange the role of e_1 and e_2. By definition of the strong product it holds that $|e| = |e_1|$ and $p_1(e) = e_1$ which implies that $p_1(x) \neq p_1(x')$ for all distinct vertices $x, x' \in e$. Thus, $p_1(x) \mapsto p_2(x)$ indeed defines a mapping $e_1 \to e_2$. Since $p_2(e) = e_2$, this mapping is surjective. Conversely, if there are edges $e_1 \in E(H_1)$ and $e_2 \in E(H_2)$ such that w.l.o.g. $p_1(x) \mapsto p_2(x)$, $x \in e$ defines a surjective mapping $e_1 \to e_2$ we can conclude that $p_1(e) = e_1$ and $p_2(e) = e_2$ and thus, in particular that $|p_1(e)| = |e_1|$. Moreover, it follows that $|e| = |p_1(e)|$, since $p_1(x) \mapsto p_2(x)$ defines a mapping and moreover, $|p_2(e)| \leq |p_1(e)| = |e_1|$, since this mapping is surjective. Hence, e satisfies the condition (ii) in the definition of the edges in the strong product and thus, $e \in E(H_1 \mathbin{\widehat{\boxtimes}} H_2)$. Finally, it is well-known, that for any two sets N, M with $|N| \geq |M|$ there are $|M|!S_{|N|,|M|}$ surjective mappings from N to M. Applying this result to every pair of edges $e_1 \in E(H_1)$ and $e_2 \in E(H_2)$ the assertion for $|\overset{\frown}{\times}|$ follows. $\square$

List of Figures

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Curriculum Vitae

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Scientific Talks

- Grid Property and Product-like Hypergraphs
  Indo-Slovenia Conference on Graph Theory and Applications, Thiruvananthapuram, India, February 22-24, 2013
- Grid property and product-like structures in hypergraphs
  28th TBI Winterseminar, Bled, Slovenia, February 10-17, 2013

- USP-relations, Equitable Partitions and Cartesian Product
  10. Herbstseminar der Bioinformatik (2012), Doubice, Czech Republic, October 2-7, 2012

- Square Property and Equitable Partitions
  Eurogiga Midterm Conference, Prague, Czech Republic, July 9-13, 2012

- How to find Cartesian Product Graphs in “Graphs with Square Property”
  27th TBI Winterseminar, Bled, Slovenia, February 12-19, 2012

- The $\mathcal{L}2$-section of hypergraphs and its application for factorizing hypergraph products
  9. Herbstseminar der Bioinformatik (2011), Vysoka Lipa, Czech Republic, September 29- October 4, 2011

- Quasi Graph Bundles
  7th Slovenian International Conference on Graph Theory, Bled, Slovenia, June 19-25, 2011

- Twisted Cartesian Products
  26th TBI Winterseminar, Bled, Slovenia, February 13-20, 2011

- Square Property
  8. Herbstseminar der Bioinformatik (2010), Vysoka Lipa, Czech Republic, October 5-10, 2010

- Hypergraph Products
  25th TBI Winterseminar, Bled, Slovenia, February 13-20, 2010

- Hypergraph Products
  7. Herbstseminar der Bioinformatik (2009), Vysoka Lipa, Czech Republic, October 21-25, 2009

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1. M. Hellmuth, T. Marc, L. Ostermeier, and P. F. Stadler. The relaxed square property. *Australas. J. Combin.*, accepted for publication, 2015.

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4. M. Hellmuth, L. Ostermeier, and M. Noll. Strong products of hypergraphs: Unique prime factorization theorems and algorithms. *Discrete Appl. Math.*, 171:60–71, 2014.

Other Publications

1. M. Hellmuth, L. Ostermeier, and P. F. Stadler. A survey on hypergraph products. *Math. Comput. Sci.*, 6:1–32, 2012.

2. M. Hellmuth, L. Ostermeier, and P.F. Stadler. Diagonalized Cartesian products of S-prime graphs are S-prime. *Discrete Math.*, 312(1):74 – 80, 2012. Algebraic Graph Theory - A Volume Dedicated to Gert Sabidussi on the Occasion of His 80th Birthday.

3. L. Ostermeier, M. Hellmuth, and P. F. Stadler. The Cartesian product of hypergraphs. *J. Graph Theory*, 70:180–196, 2012.

Selbständigkeitserklärung

Hiermit erkläre ich, die vorliegende Dissertation selbständig und ohne unzulässige fremde Hilfe angefertigt zu haben. Ich habe keine anderen als die angeführten Quellen und Hilfsmittel benutzt und sämtliche Textstellen, die wörtlich oder sinngemäßaus veröffentlichten oder unveröffentlichten Schriften entnommen wurden, und alle Angaben, die auf mündlichen Auskünften beruhen, als solche kenntlich gemacht. Ebenfalls sind alle von anderen Personen bereitgestellten Materialen oder erbrachten Dienstleistungen als solche gekennzeichnet

Leipzig, den 08. Januar 2015

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